Limited Path Percolation in Complex Networks

Eduardo López

Los Alamos Nat. Lab. LA-UR 07-0432

Outline

- Motivation. Percolation and its effects.
- Presentation of new limited path length percolation model
- Scaling theory of new model and results
- Targeted percolation, theory and results.
- Conclusions

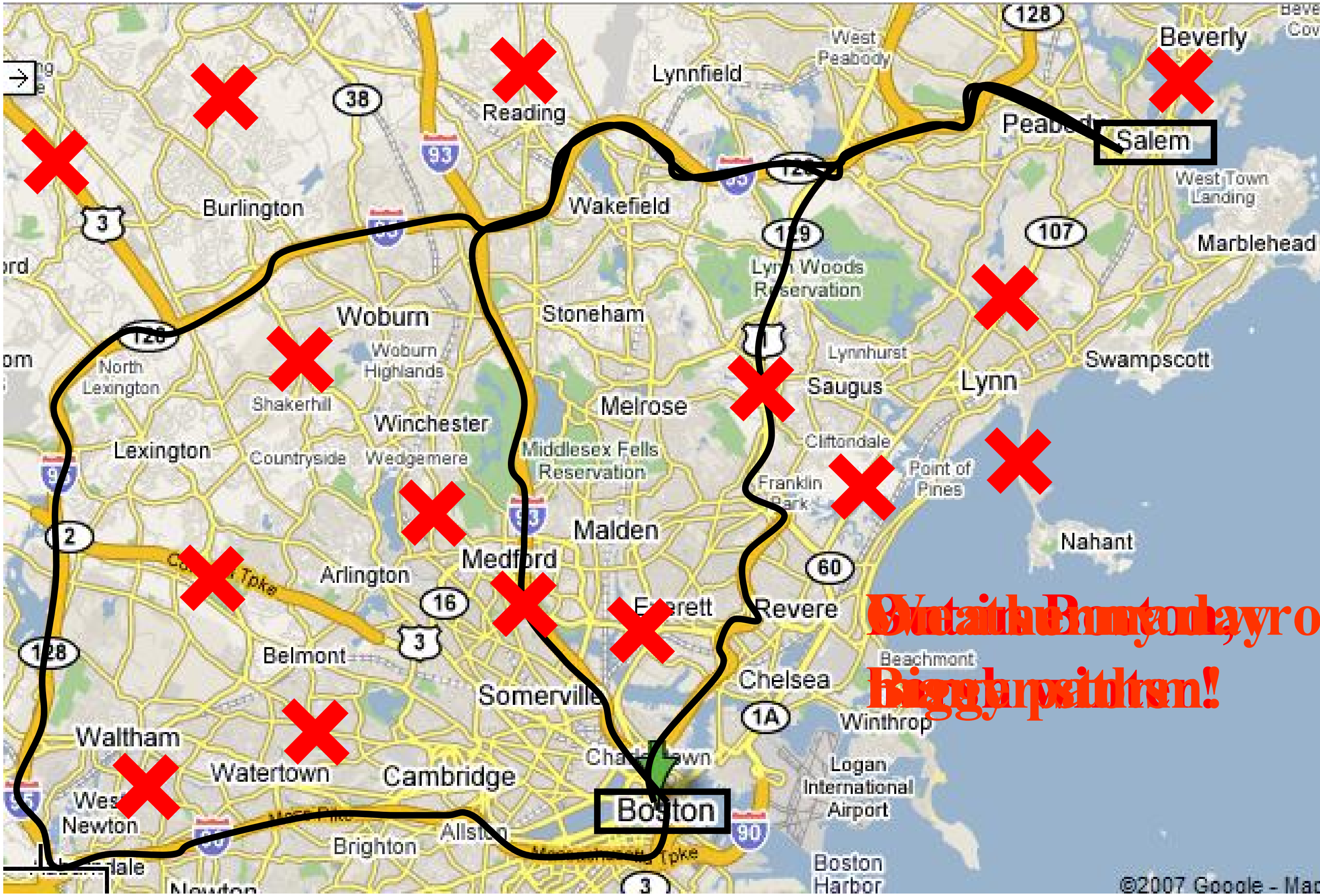
References

"Limited path length percolation in complex networks", López, Parshani, Cohen, Carmi and Havlin, Phys. Rev. Lett. (in press). cond-mat/0702691.

Collaborators

Roni Parshani	Shlomo Havlin
Shai Carmi	Reuven Cohen

Motivation: How to go from Salem to Boston?

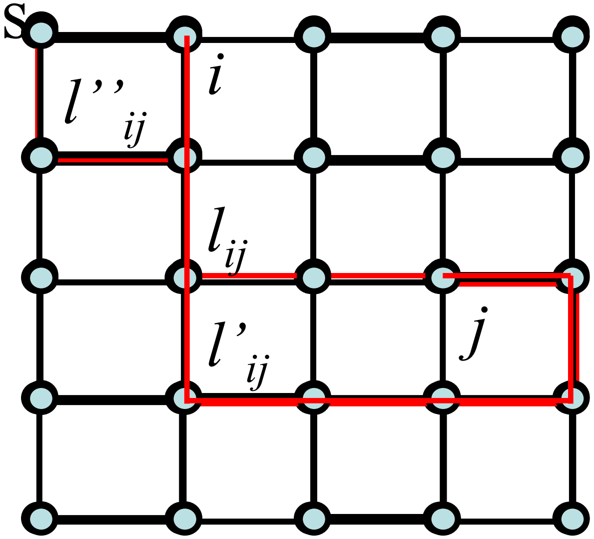


Question: How many roads need to be closed before most people cannot get to work?Answer: from Percolation theory

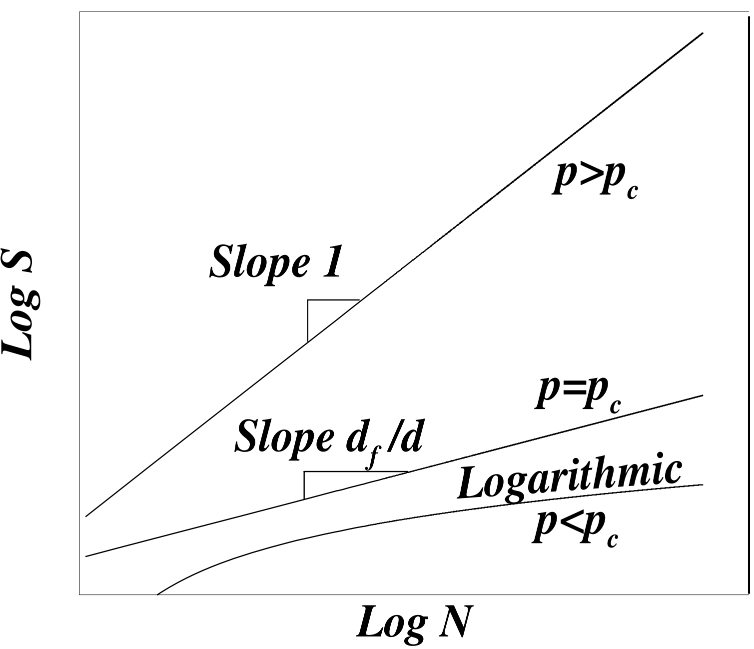
What is percolation theory?

Theory to determine connectivity in systems

p	i, j distance	$S(p)$: # connected nodes
$=1$	l_{ij}	N
<1	l'_{ij}	$P_\infty N$
	$l'_{ij} > l_{ij}$ due to removal	
$=p_c$	$l''_{ij} > l'_{ij}$	$N^{df/d}$
$<p_c$	most disconnec.	$\log N$



p: occupied fraction of links

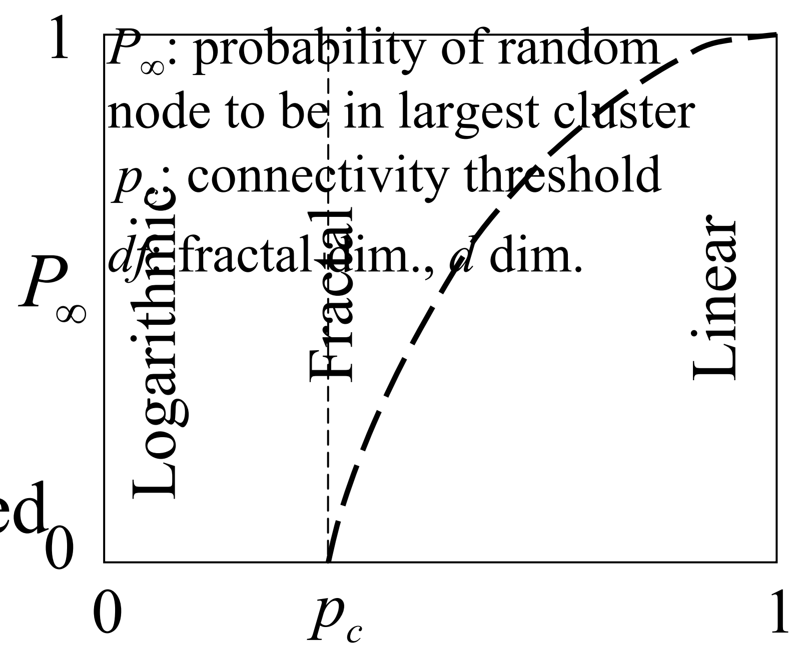


Transition:

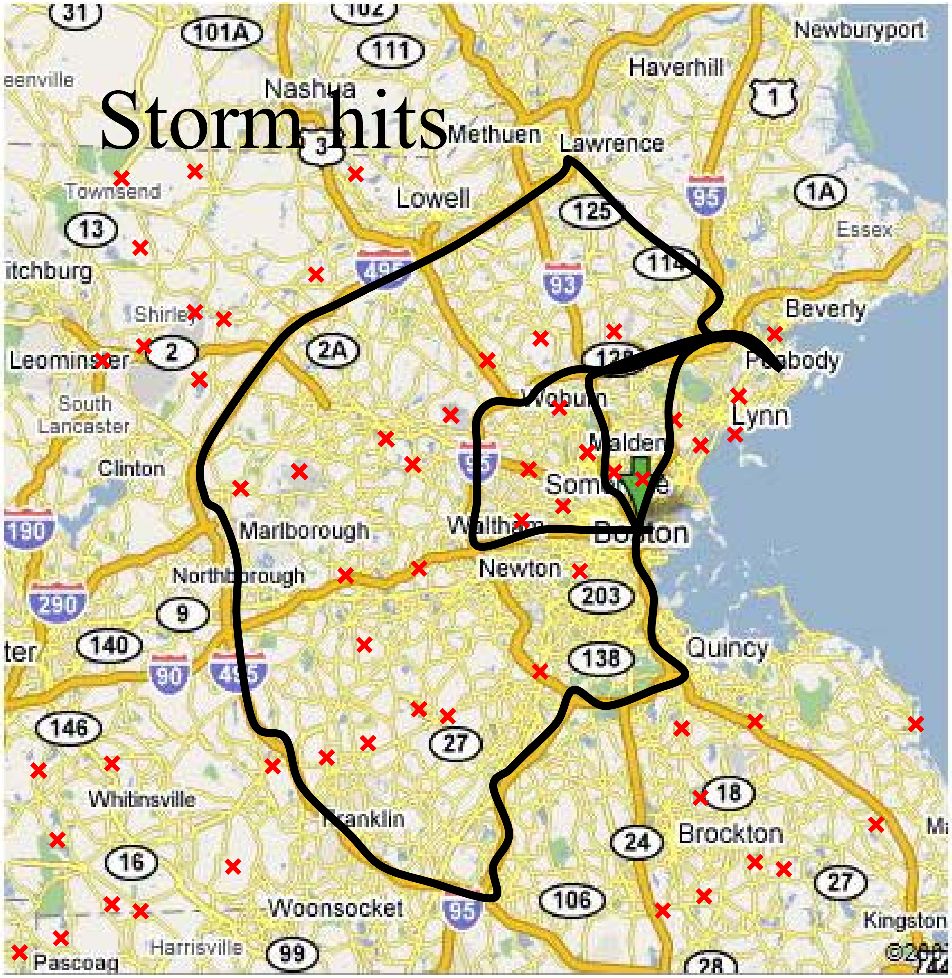
connected

↓

disconnected



Motivation: What's the problem with percolation?



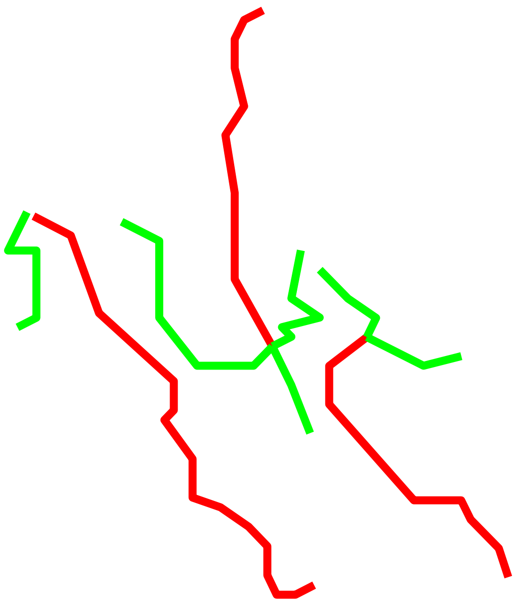
- Salem-Boston connected with any path!
- Long or short paths OK
- Percolation finds critical percentage p_c of roads needed to keep cities connected.
- Percolation increases path lengths (and time), i.e., smaller $p \Rightarrow$ longer path.
- There is practical limit to connectivity $\Leftrightarrow$ longer paths not useful.

Commute time: [illegible] min All day driving!

Answer: sometimes percolation accepts useless paths.

Social contact network

[redacted]



New percolation model applied to complex networks

- Definition of connection: i and j are connected if $l'_{ij} \leq a l_{ij}$
- Notation:

  $S_a(p)$: Largest cluster size at occupation p, length condition a

- Is there a critical occupation $p = \tilde{p}_c$ above which $S_a \sim N$?

Results: New limited path percolation transition

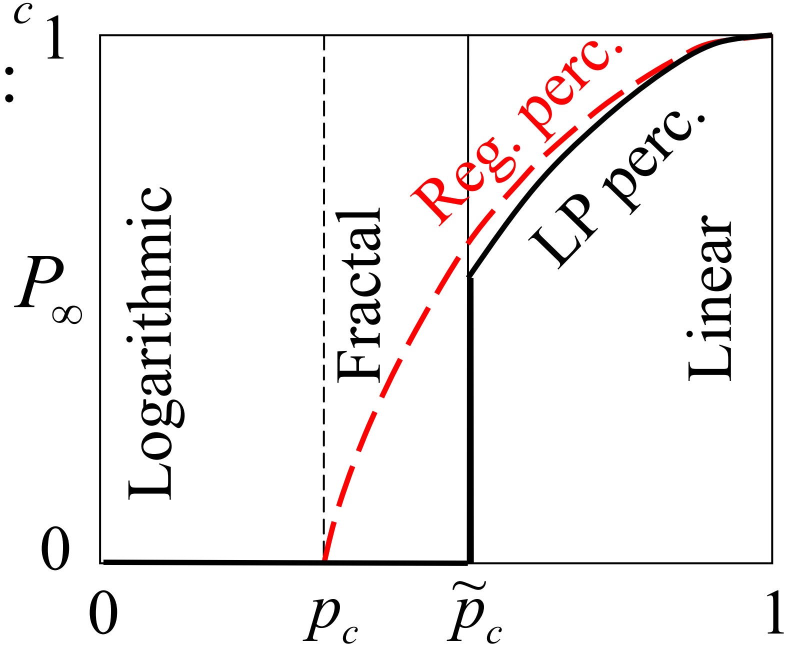
- Scaling theory
- Find new critical occupation $\tilde{p}_c > p_c$
- Critical point is now a critical range:

$$S_a \sim N^{\delta}, \delta = \delta(a,p) \left(p_c < p < \tilde{p}_c\right)$$

- Below and above range, behavior is similar to regular percolation:

$$S_a \sim \log N \left(p < p_c\right)$$

$$S_a \sim N \left(p > \tilde{p}_c\right)$$



Theory of model networks: Erdős-Rényi

- Developed in the 1960's by Erdős and Rényi. **(Publications of the Mathematical Institute of the Hungarian Academy of Sciences, 1960).**

- N nodes and each pair connected with probability ϕ.

degree of i, k_i=2

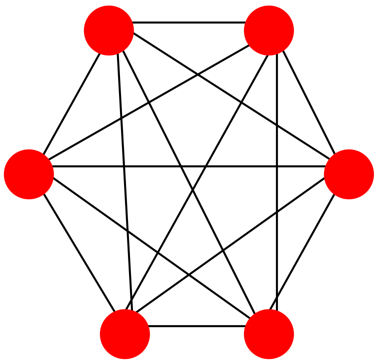
- Define k as the degree (number of links of a node), and ‹k› is average number of links per node over the network.

node i

Construction

degree of j, k_j=3

a) Complete network



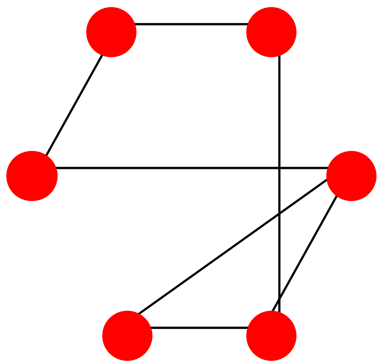
b) Annihilate links with probability

node j

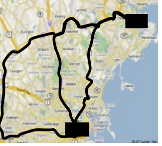
$$1-\phi$$

$$\left[\phi = \frac{\langle k \rangle}{N-1}\right]$$

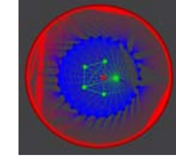
c) Realization of network



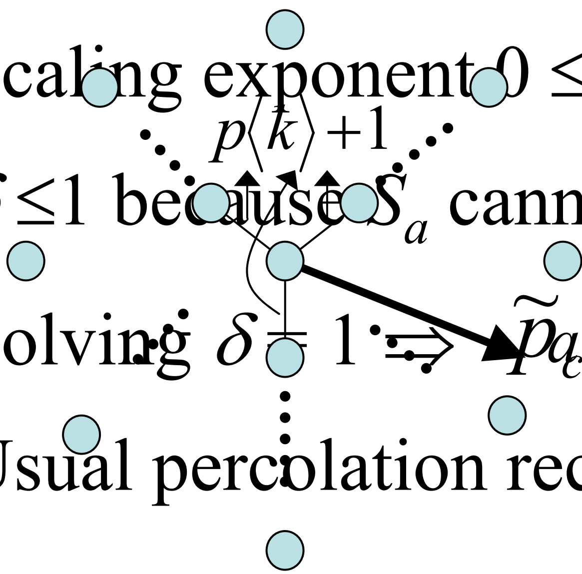
- Distribution of degree is Poisson-like (exponential) $P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$

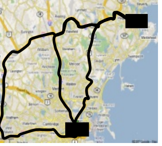


Outline of scaling theory for Limited Path Percolation Example: Erdős-Rényi

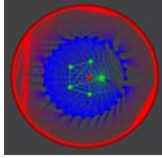


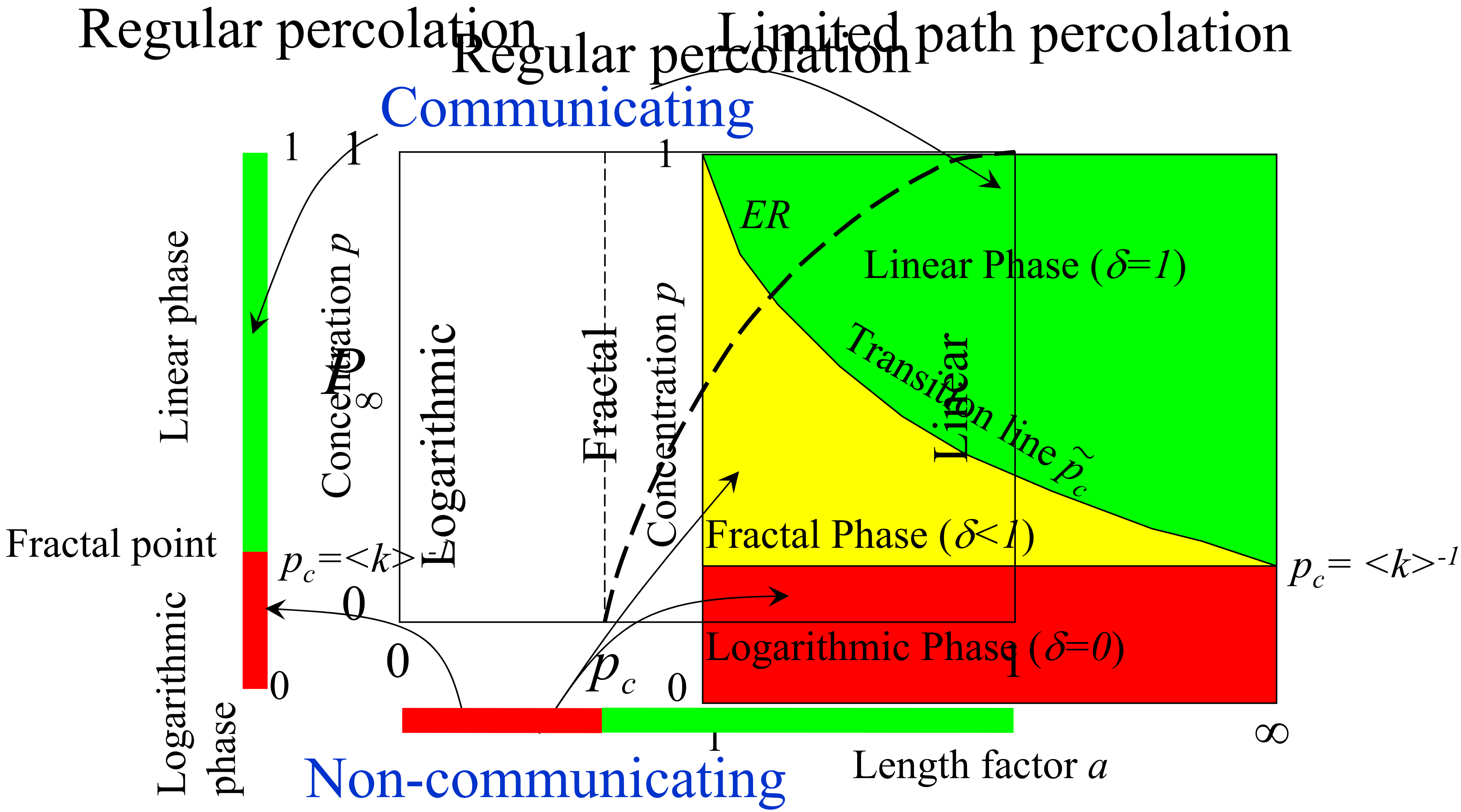
- Before percolation, typical path length $l \sim \log N / \log \langle k \rangle$
- After percolation, local structure is tree-like, with branching factor $\kappa = p\langle k \rangle + 1$
- Tree approx. $\Rightarrow S_a \sim (\kappa - 1)^l = (p\langle k \rangle)^{a \log N / \log \langle k \rangle} = N^{\delta}$
- Scaling exponent $0 \leq \delta \equiv a(1 + \log p / \log \langle k \rangle) \leq 1$
- $\delta \leq 1$ because S_a cannot exceed N
- Solving $\delta = 1 \Rightarrow \tilde{p}_c \log N = \langle k \rangle^{(1-a)/a} \log \langle k \rangle$
- Usual percolation recovered with $a \to \infty$: $\tilde{p}_c \xrightarrow[a \to \infty]{} p_c = \langle k \rangle^{-1}$



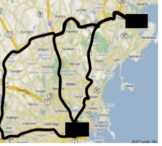


Comparison of phase diagram of regular & Limited Path Percolation (Erdős-Rényi)

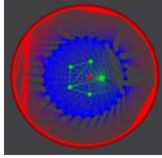




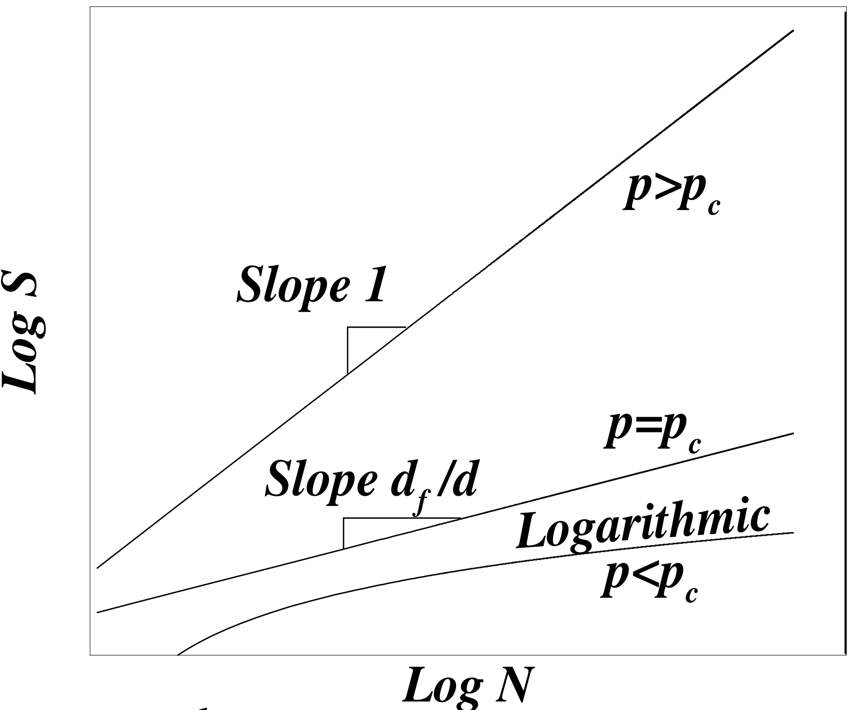
Limited path percolation predicts a larger communication threshold.



Results for $S_a \sim N^{\delta}$ (Erdős-Rényi)



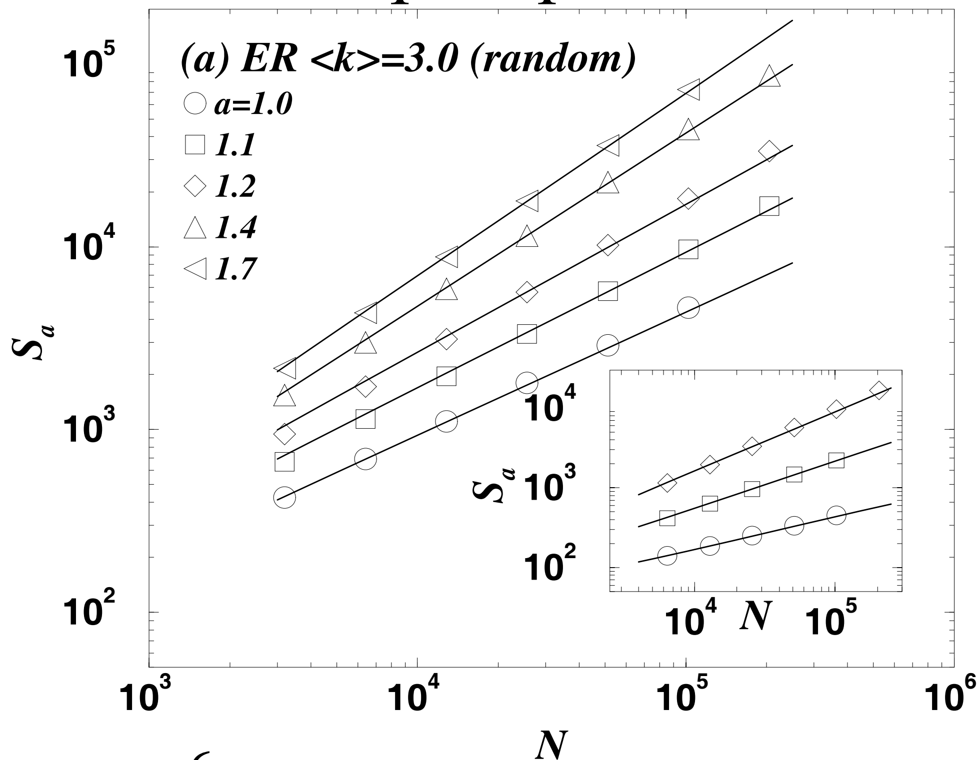
Regular Percolation



$$S \sim \begin{cases} N & (p > p_c) \\ N^{2/3} & (p = p_c) \\ \log N & (p < p_c) \end{cases}$$

$$p_c = \langle k \rangle^{-1}$$

Limited path percolation



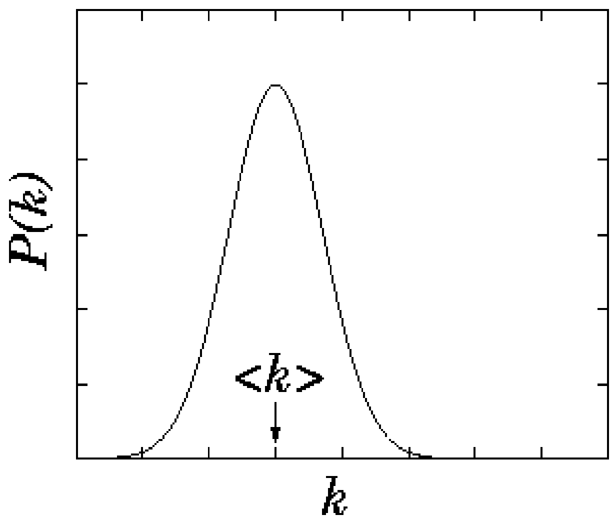
$$S \sim \begin{cases} N & (p > \tilde{p}_c) \\ N^{\delta} & (p_c \le p \le \tilde{p}_c) \\ \log N & (p < p_c) \end{cases}$$

$$\tilde{p}_c = \langle k \rangle^{(1-a)/a}$$

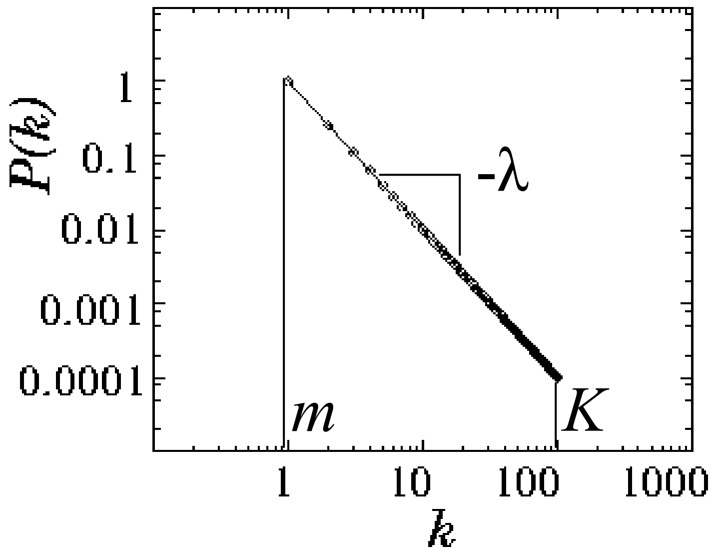
$$\delta \equiv a \log \langle pk \rangle / \log \langle k \rangle$$

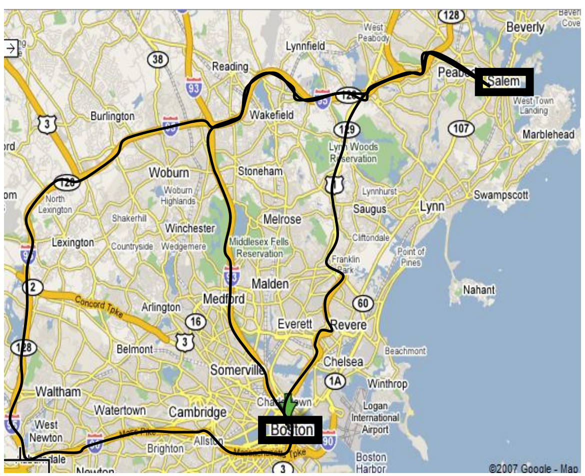
Complex Networks

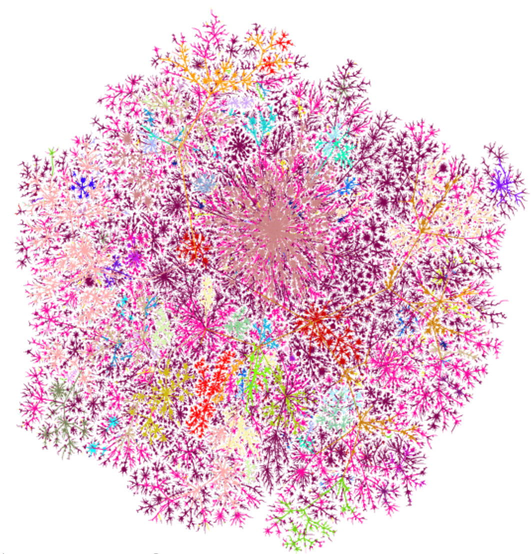
Poisson distribution



Scale-free distribution







Erdős-Rényi Network

Scale-free Network

Some basic network properties

Erdős-Rényi networks

•Narrow range of typical degree

$$\langle k\rangle - \sqrt{\langle k\rangle} \le k \le \langle k\rangle + \sqrt{\langle k\rangle}$$

•Small diameter

$$D \sim \ln N$$

Scale-free networks

•Wide range of typical degree

$$k_{\min} \le k \le k_{\min} N^{1/(\lambda-1)}$$

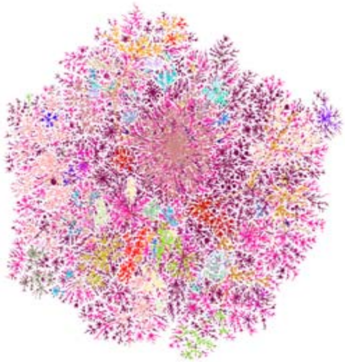
($k_{\min}$ is minimum degree)

•Small or ultra-small diameter

$$D \sim \ln(\ln N)\,[2 < \lambda < 3]$$

$$D \sim \ln N\,[\lambda > 3]$$

Scaling theory for limited path percolation on scale-free networks



- For $\lambda>3$:

$$S_a \sim N^{a[1+\log p/\log(\kappa_o-1)]}$$

$$\tilde{p}_c = (\kappa_o - 1)^{(1-a)/a}$$

- For $2<\lambda<3$:

Tree approximation invalid. Networks are ultra-small:

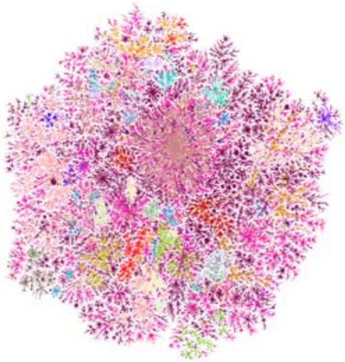
$$l \sim \log\log N / |\log(\lambda-2)|$$

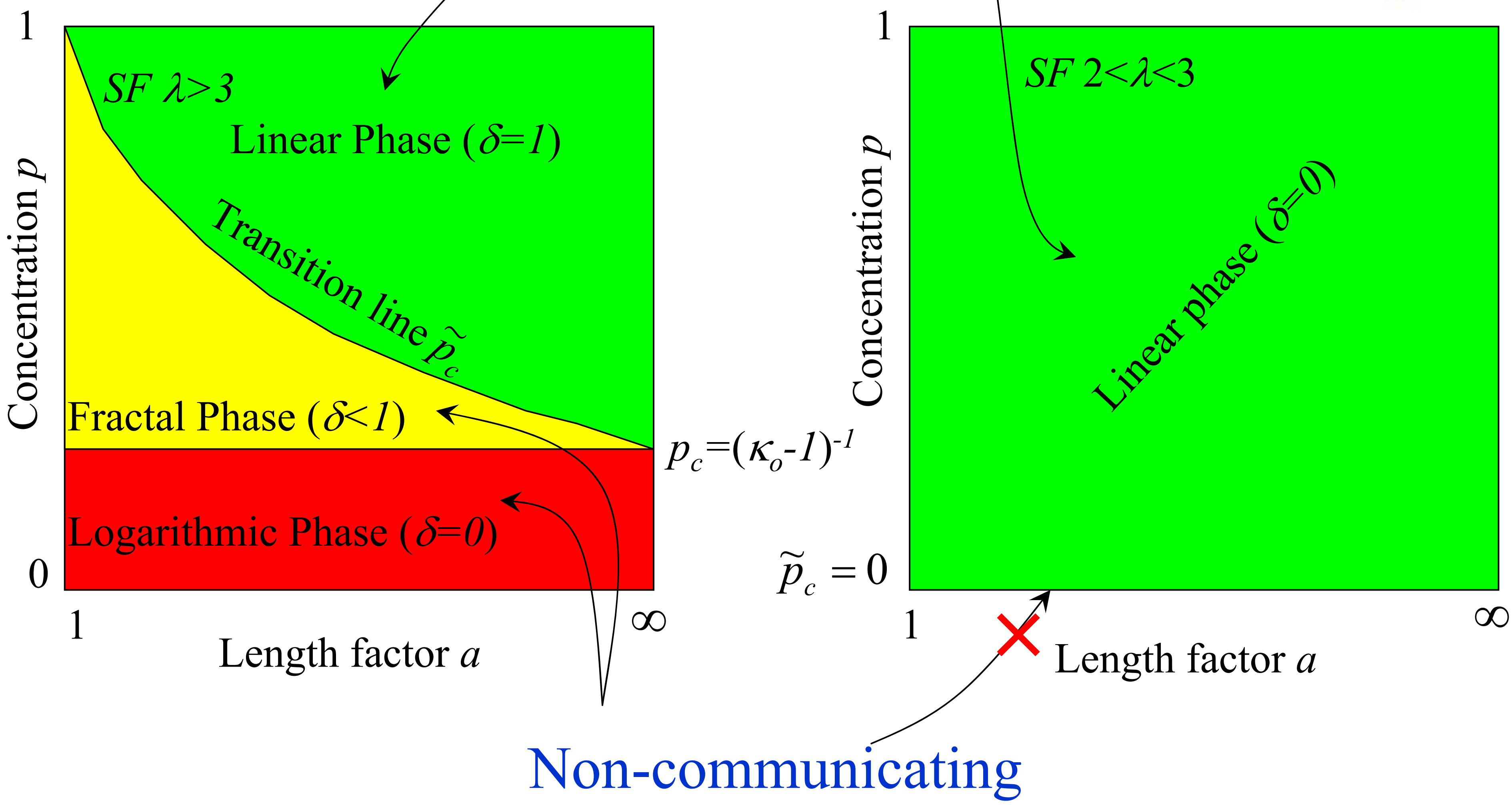
$$l' \sim \log\log P_\infty N / |\log(\lambda-2)|$$

Therefore:

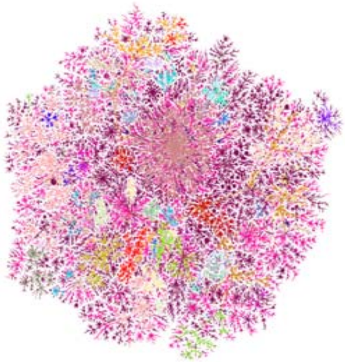
$$a = \frac{l'}{l} \sim \frac{\log\log P_\infty N}{\log\log N} \underset{N\to\infty}{\longrightarrow} 1$$

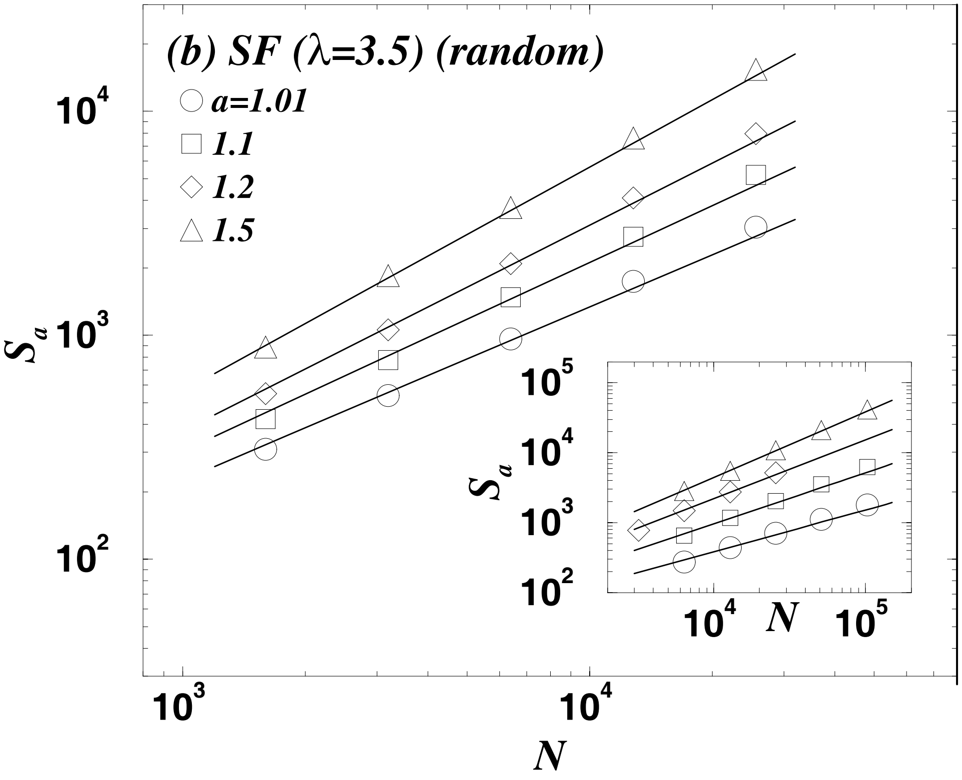
Phase Diagram of Limited Path Percolation on scale-free networks

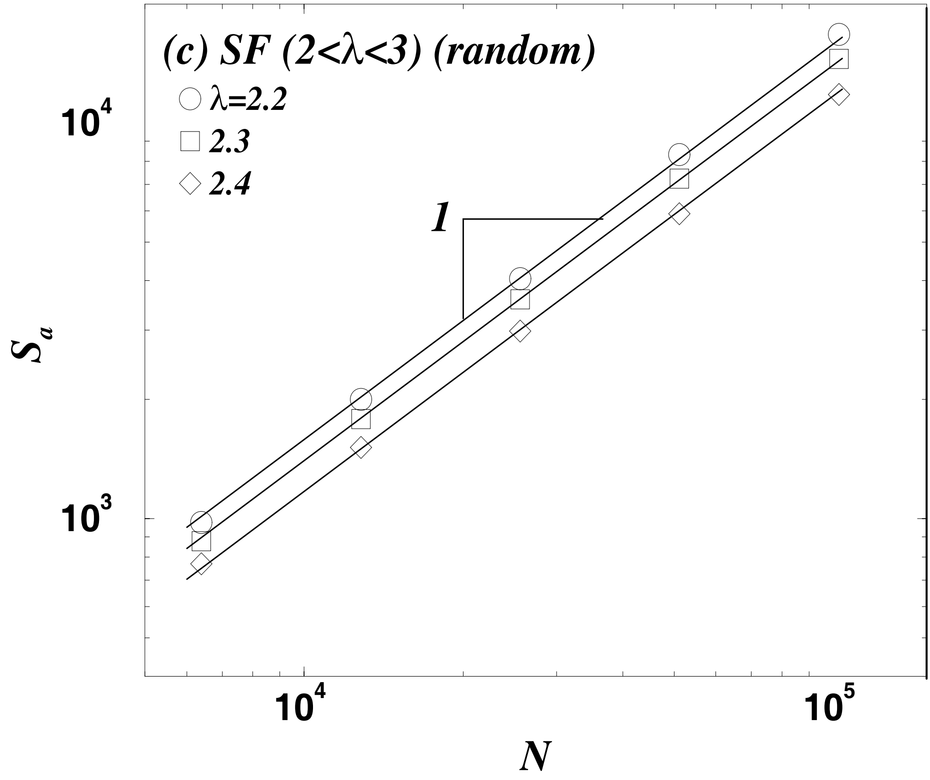




Results for $S_a \sim N^\delta$ Scale-free



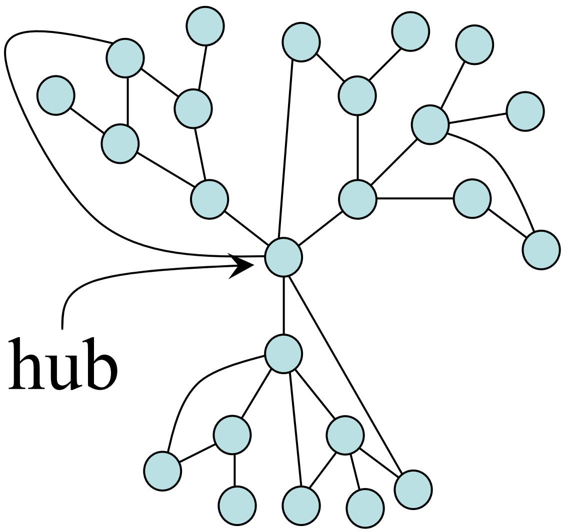




Targeted attacks on scale-free networks

- Scale-free networks have sensitive nodes (hubs) with large k.
- Examples: Airline hubs, central communication nodes, disease super-spreaders.

Model for targeted percolation



- p: fraction of lowest degree nodes present.
- In targeted percolation (no length restriction) p_c is large:

p_c=1 ($\lambda \to 2$)

p_c close to 1 ($\lambda > 2$)

Network falls apart with few node removals.

Question: What happens for limited path percolation?

Scaling theory for limited path targeted percolation on scale-free networks

- For $\lambda>3$:

$$S_a \sim N^{a\log(\kappa-1)/\log(\kappa_o-1)}$$

$$\tilde{p}_c = \tilde{p}_c(a,\kappa,\kappa_o)$$

- For $2<\lambda<3$:

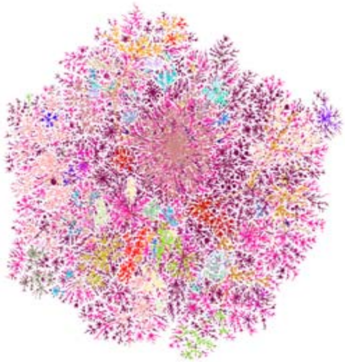
Tree approximation valid again after percolation:

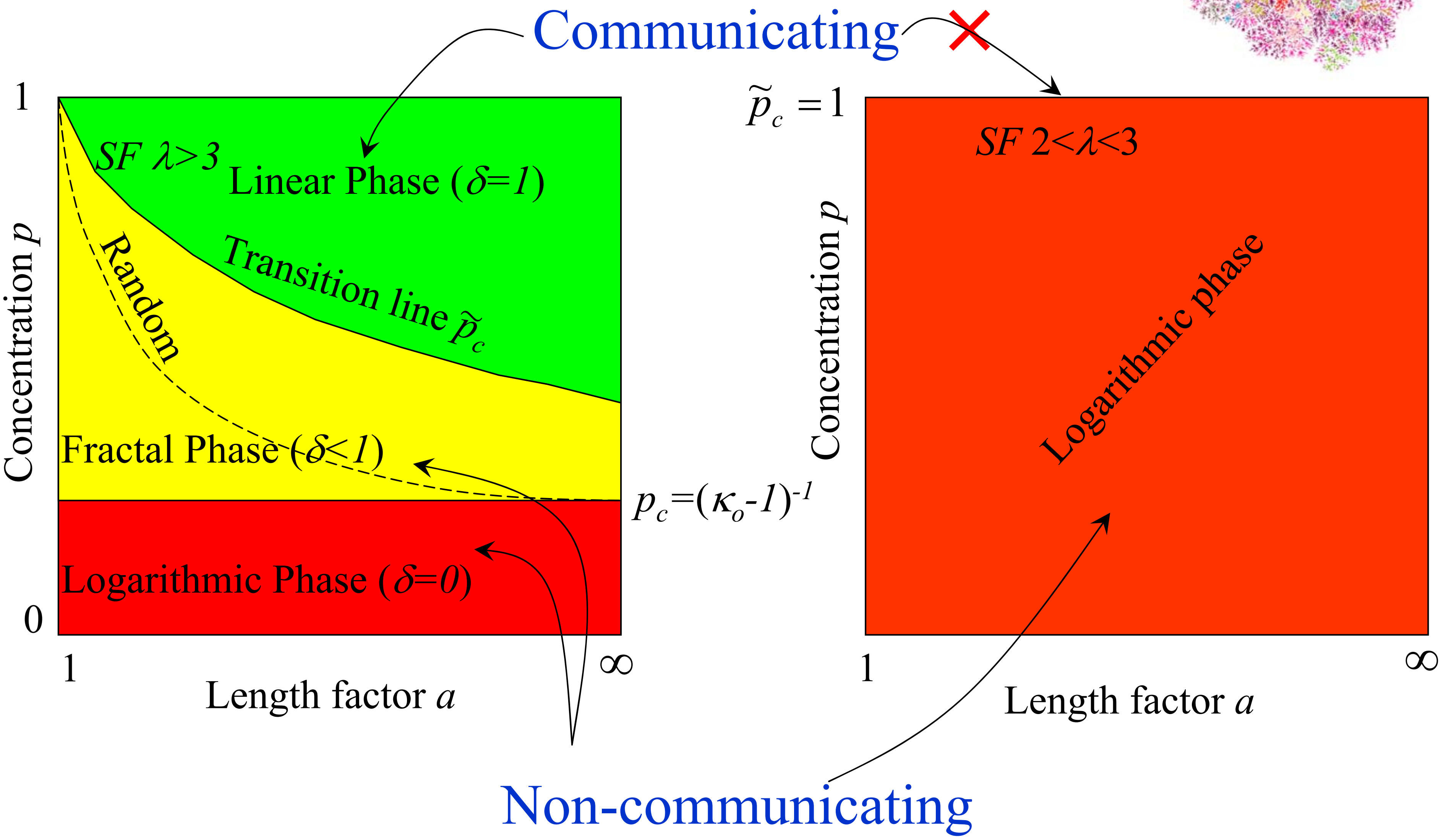
$$S_a \sim (\log N)^{2a\log(\kappa-1)/|\log(\lambda-2)|}$$

Any finite *a* fails to produce transition to linear phase:

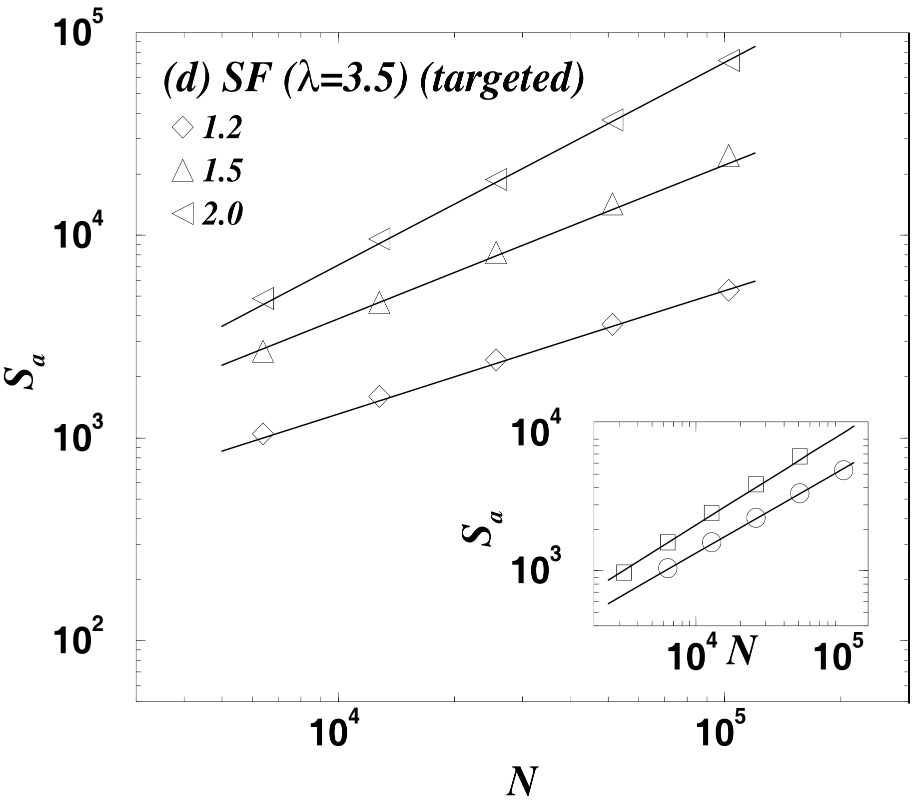
$$\tilde{p}_c = 1$$

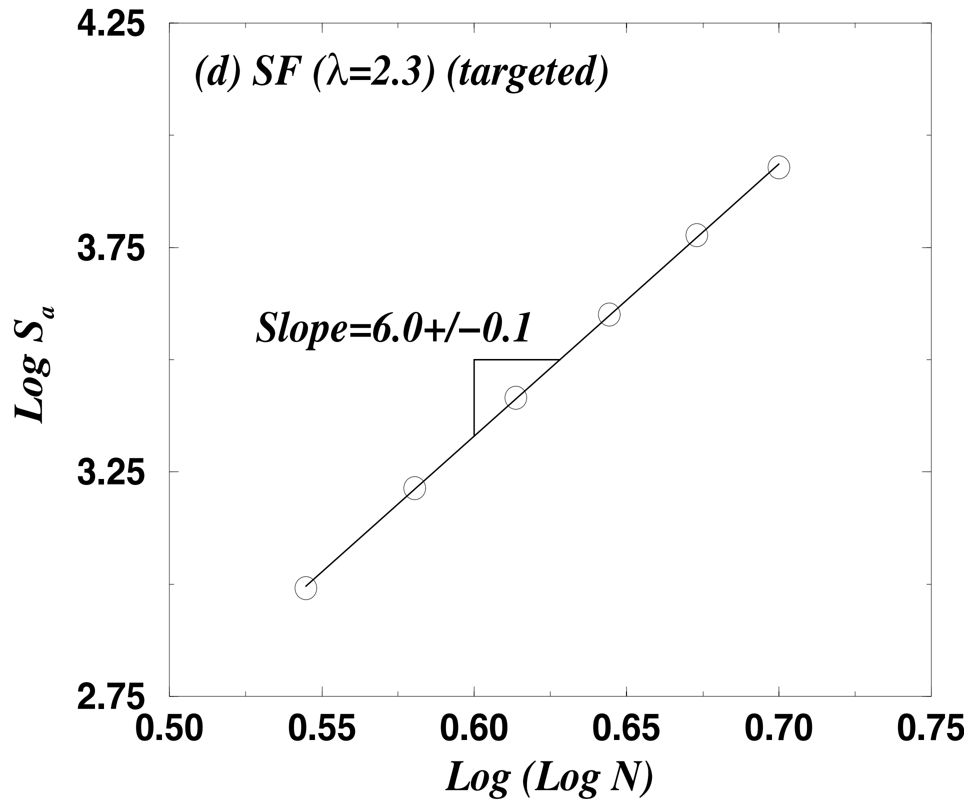
Phase Diagram of Limited Path Percolation Scale-free targeted removal





Results for $S_a \sim N^{\delta}$
Scale-free targeted removal





Differences in Limited Path Percolation due to network structure and removal method at $p_c \le p \le \tilde{p}_c$

Random removal

Quantity	Erdős-Rényi	Scale-free (λ>3)	Scale-free (2≤λ≤3)
$\tilde{p}_c$	$\langle k \rangle^{(1-a)/a}$	$(\kappa_o - 1)^{(1-a)/a}$	0
$\boldsymbol{\delta}$	$a\left(1+\frac{\log p}{\log \langle k \rangle}\right)$	$a\left(1+\frac{\log p}{\log(\kappa_o - 1)}\right)$	1
$\boldsymbol{S_a}$	$\boldsymbol{N^{\delta}}$	$\boldsymbol{N^{\delta}}$	$\boldsymbol{N^{\delta}}$

Targeted removal

Quantity	Erdős-Rényi	Scale-free (λ>3)	Scale-free (2≤λ≤3)
$\tilde{p}_c$	-	$\tilde{p}_c(a, \kappa, \kappa_o)$	1
$\boldsymbol{\delta}$	-	$a\frac{\log(\kappa - 1)}{\log(\kappa_o - 1)}$	$2a\frac{\log(\kappa - 1)}{\lvert\log(\lambda - 2)\rvert}$
$\boldsymbol{S_a}$	-	$\boldsymbol{N^{\delta}}$	$\boldsymbol{(\log N)^{\delta}}$

Transition

Transition

No Transition

Scaling function for S_a

- For Erdös-Rényi, and scale-free $\lambda > 3$ with random and targeted removal, there are two phases above and below $\tilde{p}_c$

- Therefore:

$$S_a \sim c(p) N^{\delta} f\left(\frac{P_\infty N}{c(p) N^{\delta}}\right) \quad f(x) \sim \begin{cases} x, & x << 1 \\ cnst., & x >> 1 \end{cases}$$

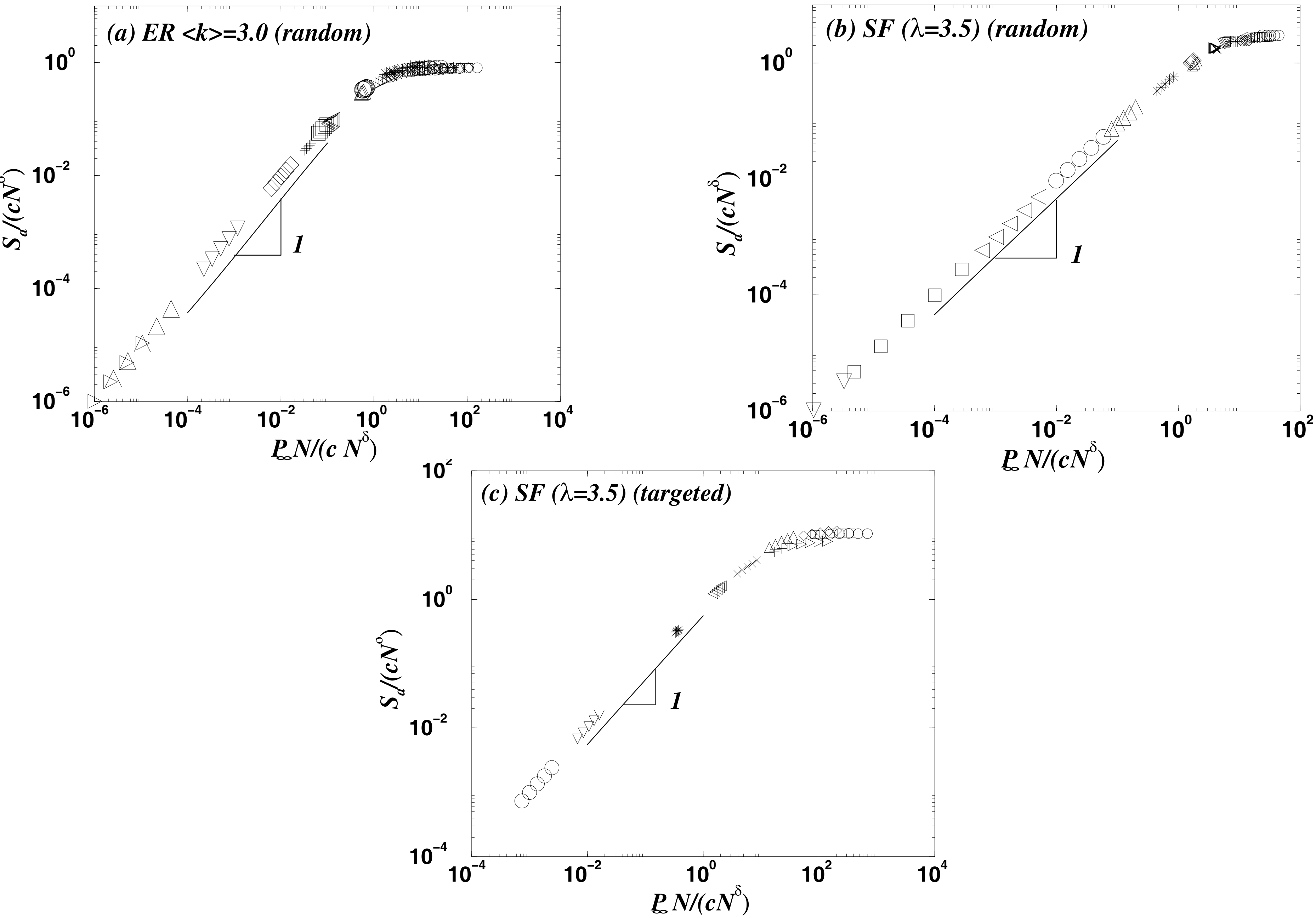
$$\mathbf{c(p) \equiv c_o \, [p \, (\kappa_o - 1) + 1] / [p(\kappa_o - 1) - 1]}$$

- Two limits:

i) $S_a \sim c(p) N^{\delta}$ $\qquad (p_c < p < \tilde{p}_c)$

ii) $S_a \sim P_\infty N$ $\qquad (\tilde{p}_c < p \leq 1)$

Results for scaling of S_a



Conclusions

• We define a new percolation model which takes into account the length restriction of useful paths.

• This model is important in real-world applications such as epidemics, data transfer, and transportation.

• We find a new percolation transition at $\tilde{p}_c = (\kappa_0 - 1)^{(1-a)/a} > p_c$ which implies when lengths are constrained, more connections are necessary to percolate. Transition preserves path length scaling.

• We encounter two typical phases: i) power-law with $S_a \sim N^{\delta}$, and ii) a linear phase $S_a \sim N$.

Conclusions

• We define a new percolation model which takes into account the length restriction of useful paths.

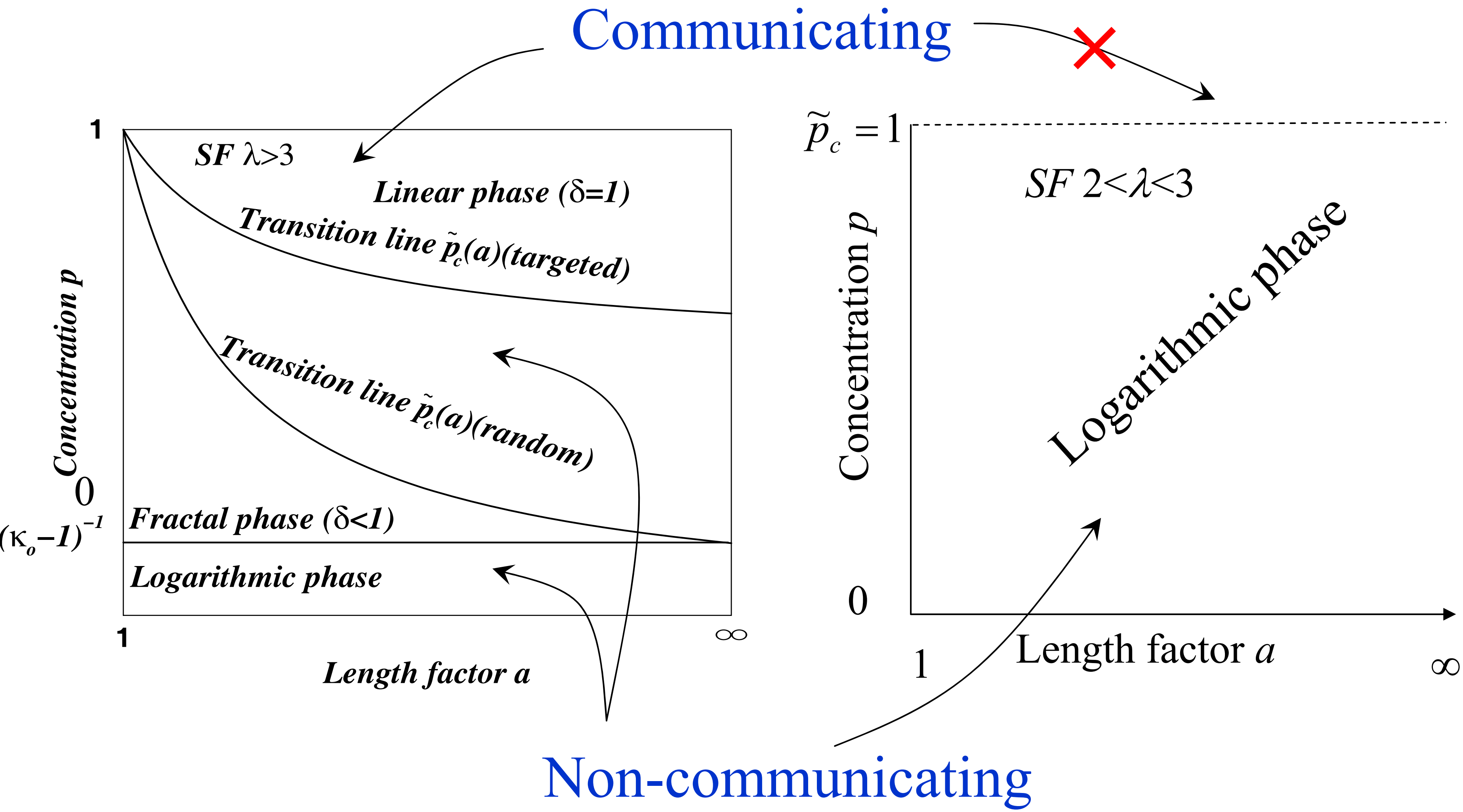
• This model is important in real-world applications such as epidemics, data transfer, and transportation.

• We find a new percolation transition at $\tilde{p}_c = (\kappa_0 - 1)^{(1-a)/a} > p_c$ which implies when lengths are constrained, more connections are necessary to percolate. Transition preserves path length scaling.

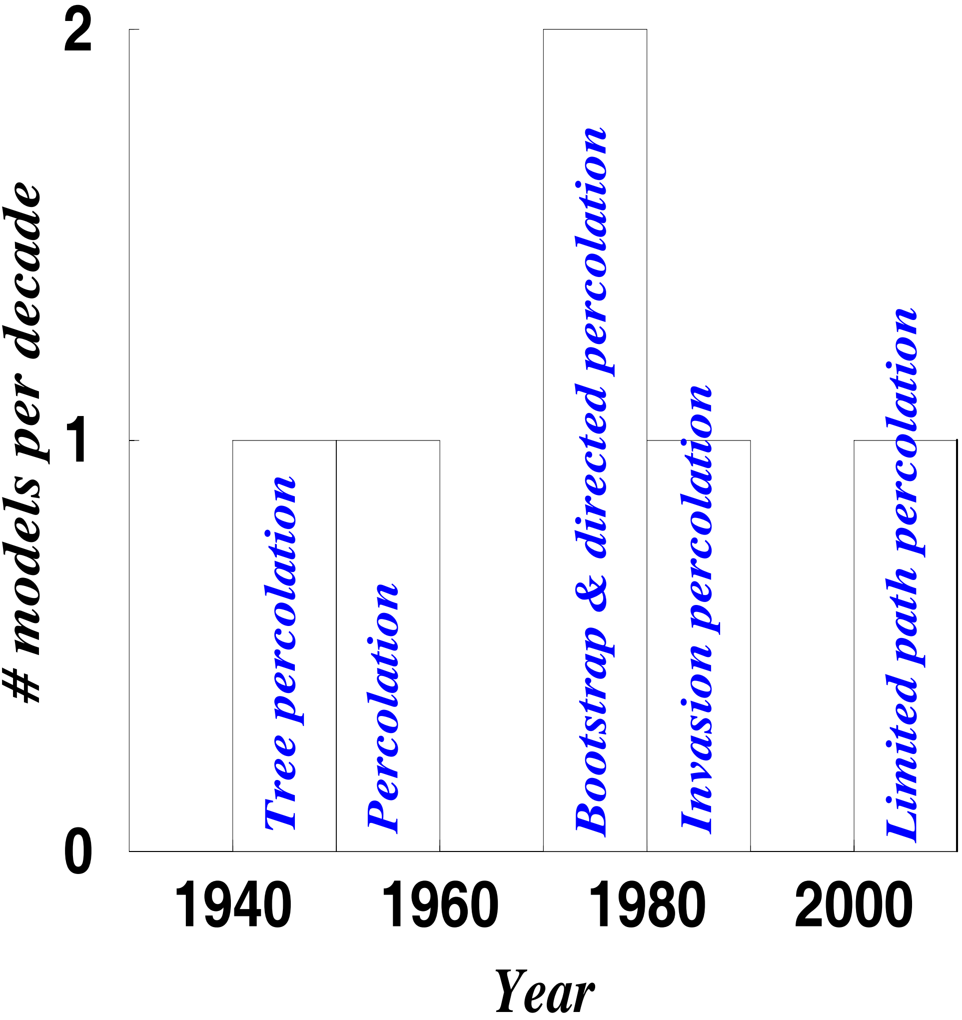
• We encounter two typical phases: i) power-law with $S_a \sim N^{\delta}$, and ii) a linear phase $S_a \sim N$.

• Few models of percolation exist. Our model is an innovative new approach to percolation with great opportunities for research.

Phase Diagram of Limited Path Percolation
Scale-free targeted removal



Timeline of percolation theory



Tree percolation
Gelation or how the egg hardens:
Flory(1941) and Stockmayer(1943).

Percolation
Flow through a random medium:
Broadbent and Hammersley(1957).

Directed percolation
Steady state chemical reactions:
Schlögl (1972).

Bootstrap percolation
Ferromagnets:
Pollak and Reiss (1975).

Invasion percolation
Displacement of fluid by another:
Wilkinson and Willemsen (1983).

Limited path percolation
Communications and epidemics:
López et al. (2007).

Molloy-Reed Algorithm for scale-free Networks

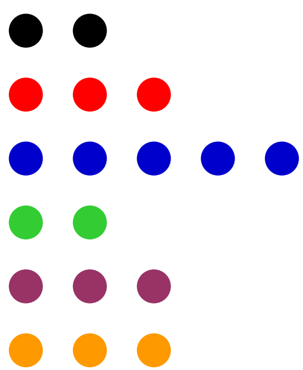
Create network with pre-specified degree distribution *P(k)*

Example:

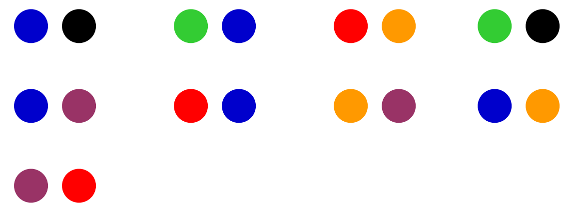
1) Generate set of nodes with pre-specified degree distribution form $P(k) \sim k^{-\lambda}$

Degree: 2 3 5 2 3 3

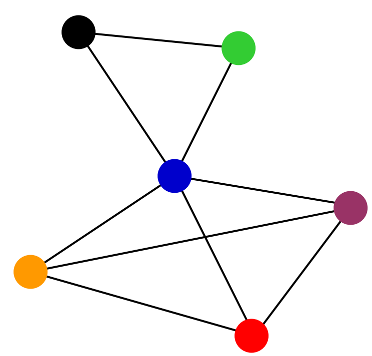
2) Make k_i copies of node i:



3) Randomly pair copies excluding self-loops and double connections:



4) Connect network:



Theory: Properties of scale-free networks

- Network size with branching factor κ_o:

  $\sim (\kappa_o\text{-}1)^l$ (λ>3); variable (2<λ<3)

- Branching factor:

  $\kappa_o = \langle k^2 \rangle / \langle k \rangle$ =cons. (λ>3); incres. (2<λ<3)

- Typical distance l between nodes:

$$l \sim \frac{\log N}{\log(\kappa_o - 1)} (\lambda > 3); \frac{\log \log N}{|\log(\lambda - 2)|} (2 < \lambda < 3)$$

- Percolation thresholds:

  $p_c = (\kappa_o\text{-}1)^{-1}$ (λ>3); 0 (2<λ<3)

- Nodes connected at $p=p_c$:

  $S \sim N^{(\lambda-3)/(\lambda-1)}$ (λ>3); N (2<λ<3)

- Branching factor at occupation p:

$$\kappa - 1 = p(\kappa_o - 1) \text{ for } (\lambda > 3)$$

Summary of theoretical results

Erdös-Rényi

$$\tilde{p}_c = \langle k \rangle^{(1-a)/a}, \; S_a \sim N^{\delta}, \quad \delta = a\left(1 + \frac{\log p}{\log\langle k \rangle}\right)$$

Scale-free (λ>3)

$$\tilde{p}_c = (\kappa_o - 1)^{(1-a)/a}, \; S_a \sim N^{\delta}, \quad \delta = a\left(1 + \frac{\log p}{\log(\kappa_o - 1)}\right)$$

Scale-free (2<λ<3)

$$\tilde{p}_c = 0, \, S_a \sim N$$

Summary of theoretical results

Targeted removal on scale-free networks

$\lambda > 3$

$$\tilde{p}_c = \tilde{p}_c(a, \kappa, \kappa_o),\, S_a \sim N^{\delta},\, \delta = a\frac{\log(\kappa - 1)}{\log(\kappa_o - 1)}$$

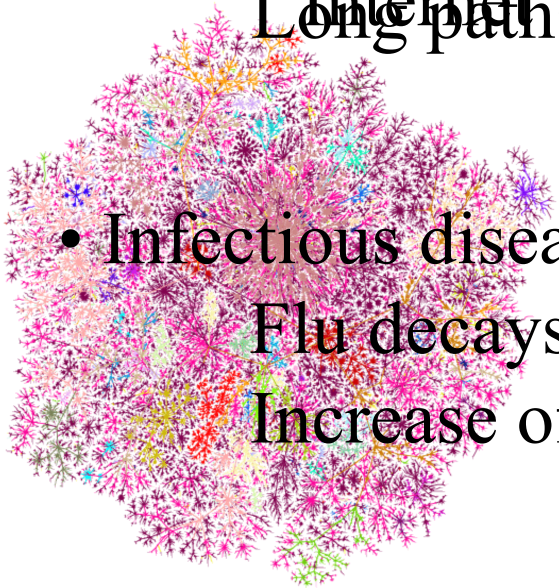
$2 < \lambda < 3$

$$\tilde{p}_c = 1,\, S_a \sim (\log N)^{\delta},\, \delta = 2a\frac{\log(\kappa - 1)}{\left|\log(\lambda - 2)\right|}$$

Motivation: Where else does percolation fail?

- Communications such as data packet routing:

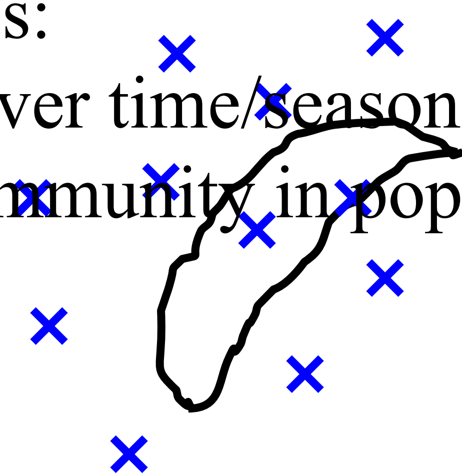
  Long paths ineffective.



- Infectious diseases:

  Flu decays over time/season.

  Increase of immunity in population.



- Transportation:

  Long commute times prohibitive.

- Other important extensions like path cost considerations.

Rerouting makes sense if new path is short

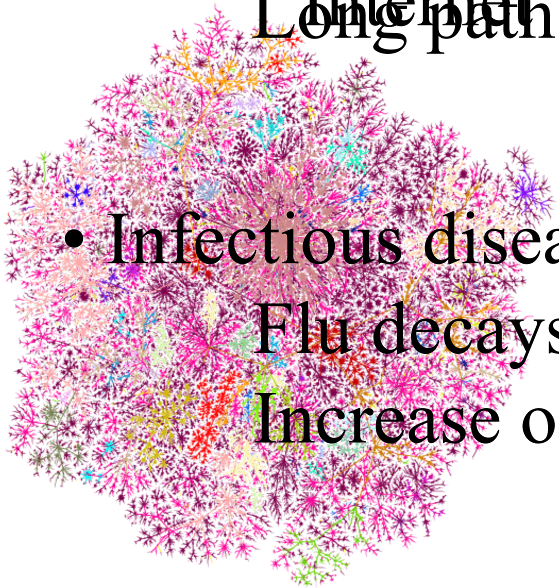
Long paths compound error + reduce performance + security

Motivation: Where else does percolation fail?

- Communications such as data packet routing:

  Long paths ineffective.

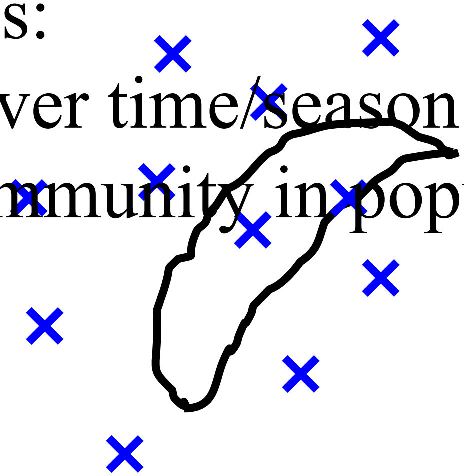
Internet



- Infectious diseases:

  Flu decays over time/season.

  Increase of immunity in population.



- Transportation:

  Long commute times prohibitive.

Communication problems require data rerouting

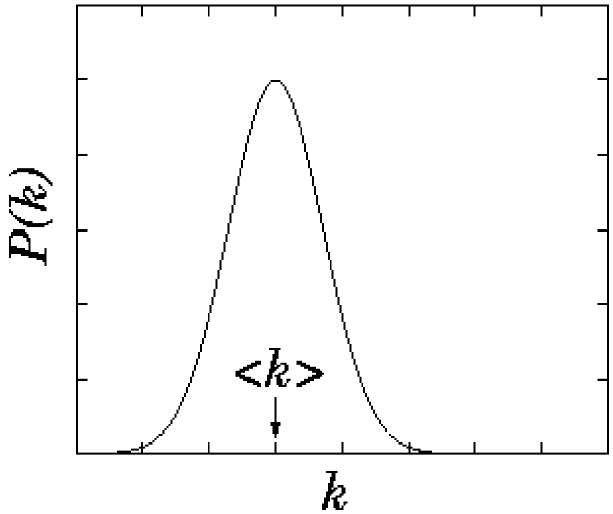
- Other important extensions like path cost considerations.

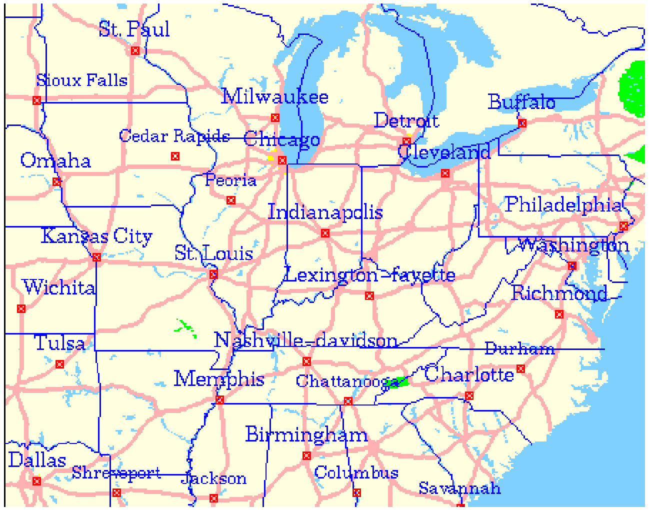
Rerouting makes sense if new path is short

Long paths compound error + reduce performance + security

Complex Networks

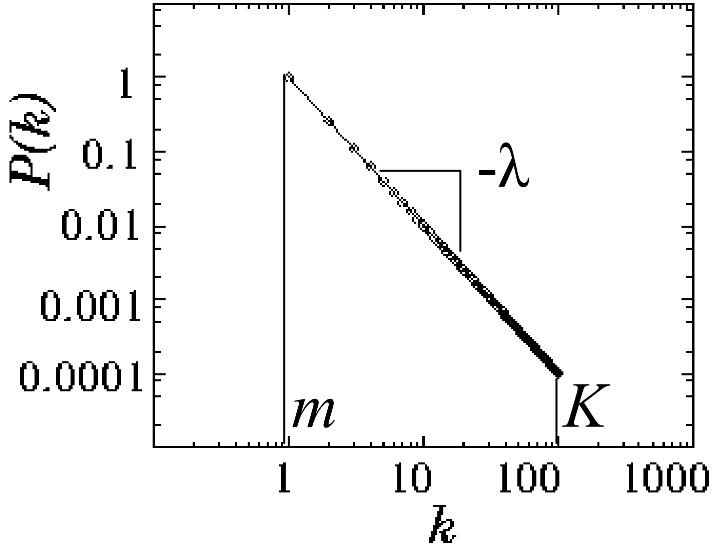
Poisson distribution

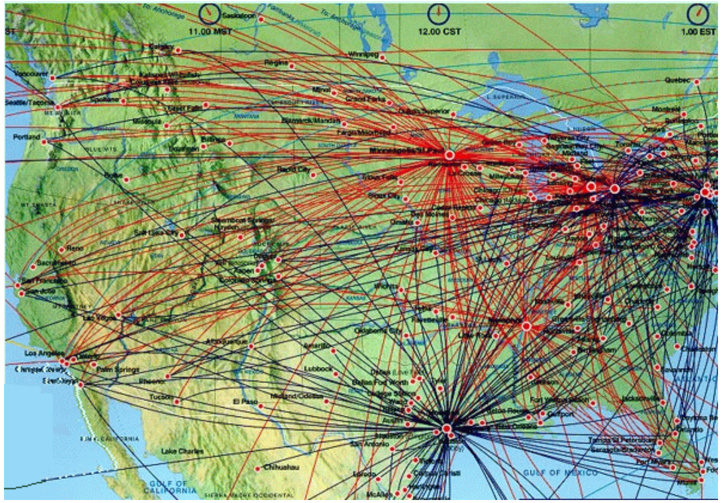




Erdős-Rényi Network

Scale-free distribution



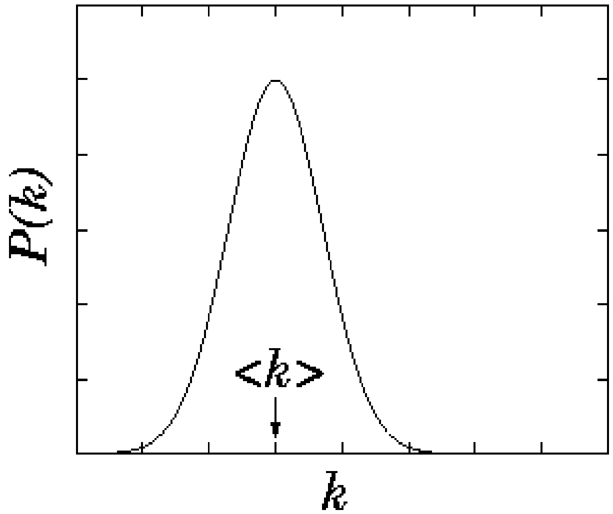


Scale-free Network

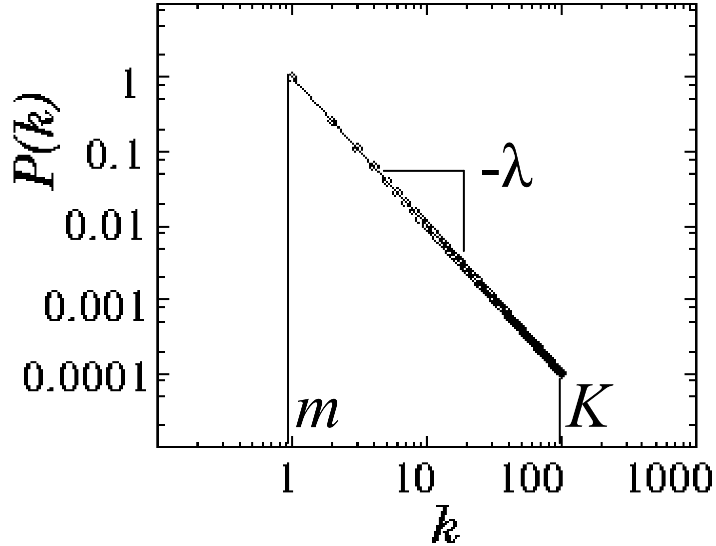
Salem
Boston
Beverly
Lynnfield
West Peabody
Reading
Burlington
Wakefield
Woburn
Stoneham
Lynn Woods Reservation
Marblehead
Swampscott
Lynn
Saugus
Lynnhurst
Melrose
Woburn Highlands
North Lexington
Shakerhill
Winchester
Lexington
Countryside
Wedgemere
Middlesex Fells Reservation
Cliftondale
Point of Pines
Malden
Medford
Arlington
Concord Tpke
Nahant
Everett
Revere
Belmont
Somerville
Chelsea
Beachmont
Winthrop
Waltham
Watertown
Cambridge
West Newton
Logan International Airport
Brighton
Boston Harbor
West Town Landing
©2007 Google - Map

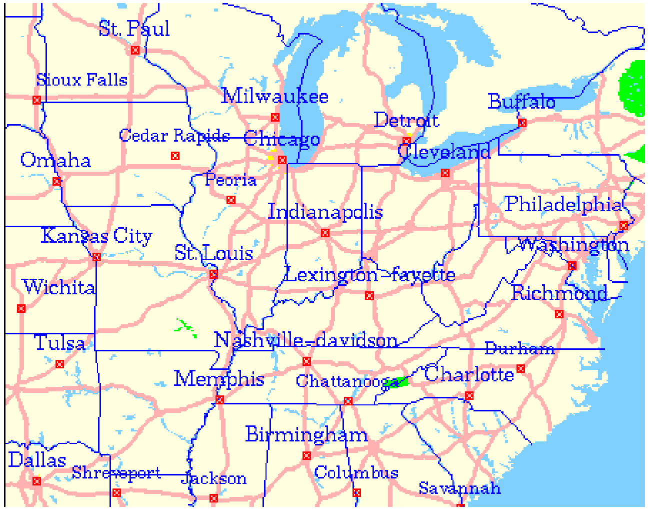
Complex Networks

Poisson distribution

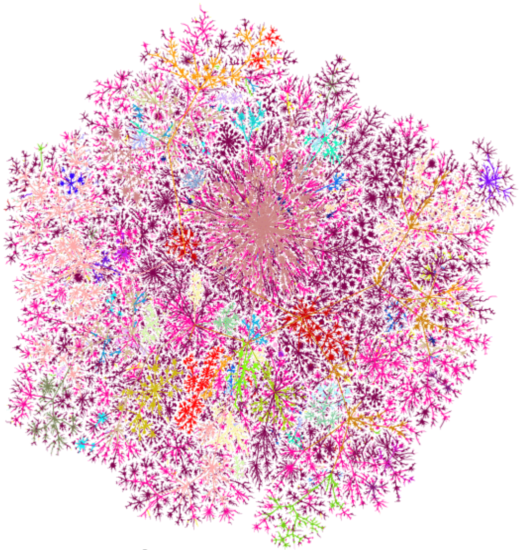


Scale-free distribution





Erdős-Rényi Network

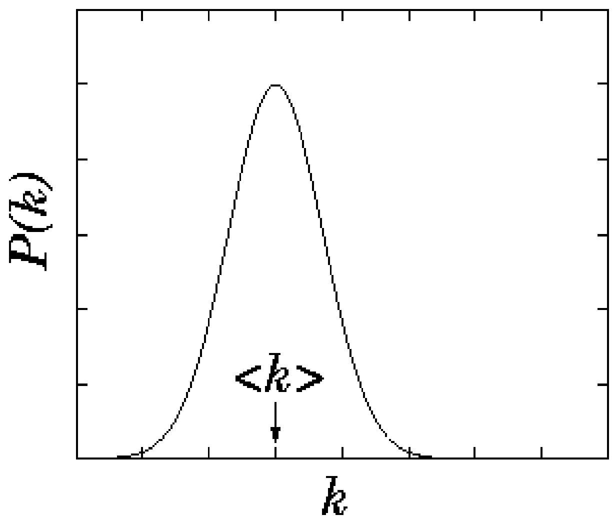


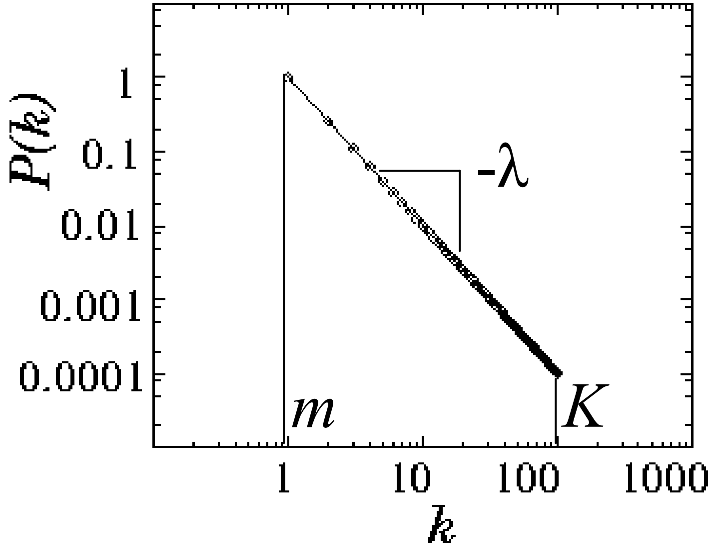
Scale-free Network

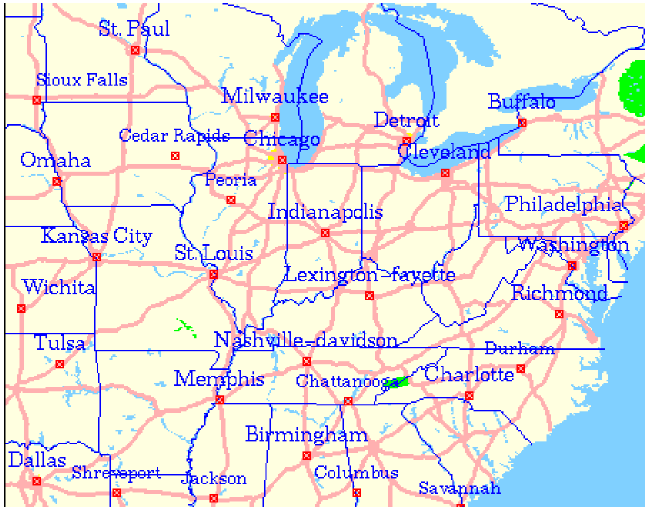
Complex Networks

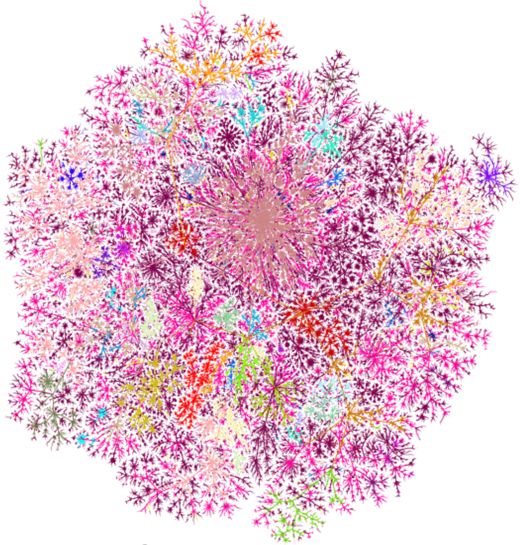
Poisson distribution

Scale-free distribution





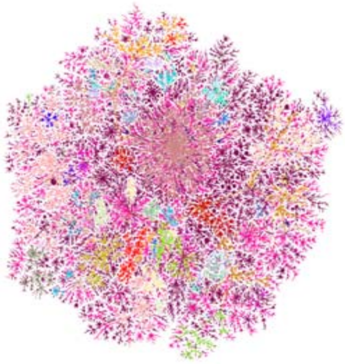


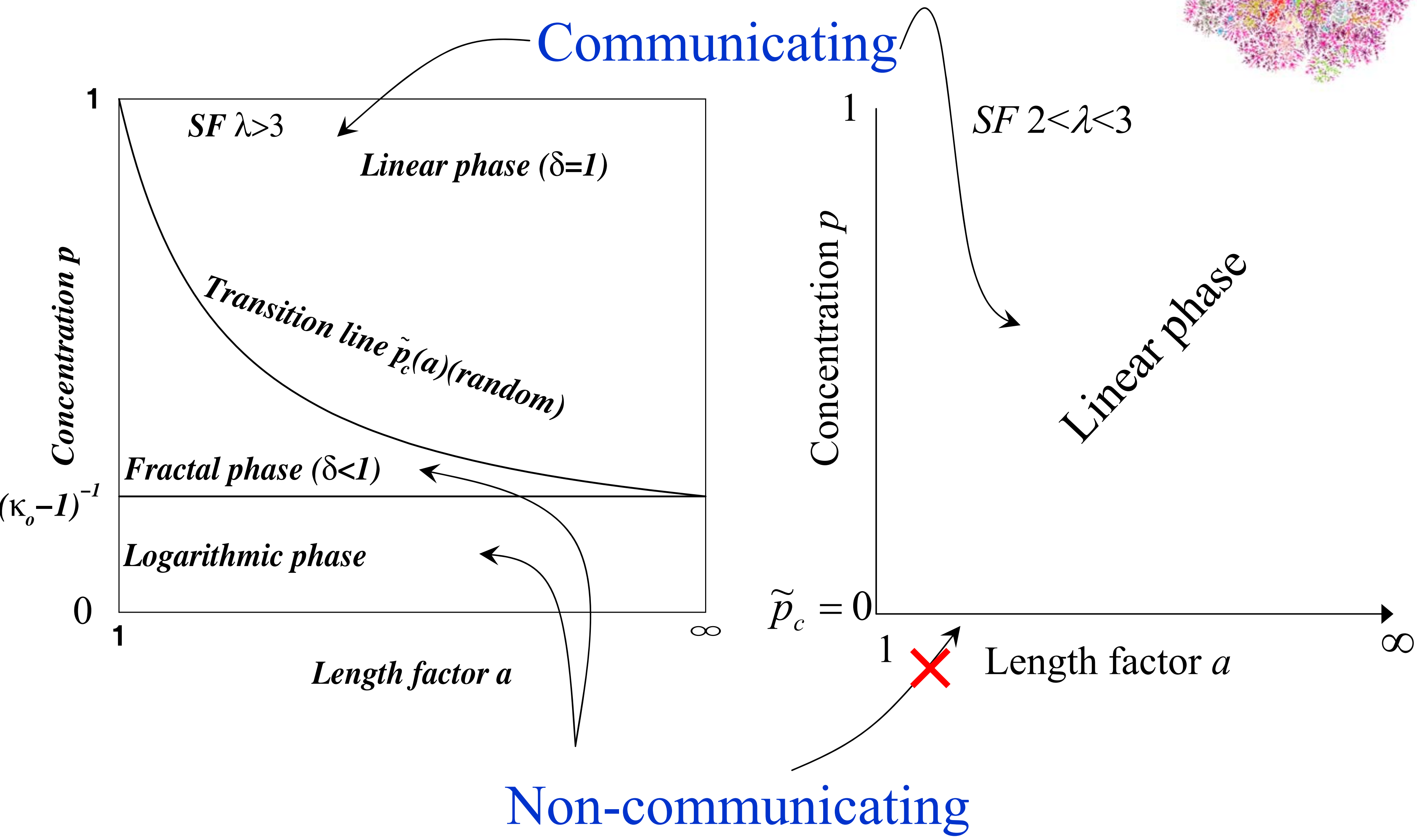


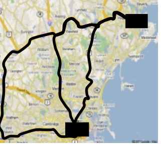
Erdős-Rényi Network

Scale-free Network

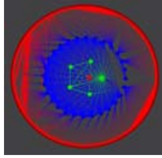
Phase Diagram of Limited Path Percolation Scale-free



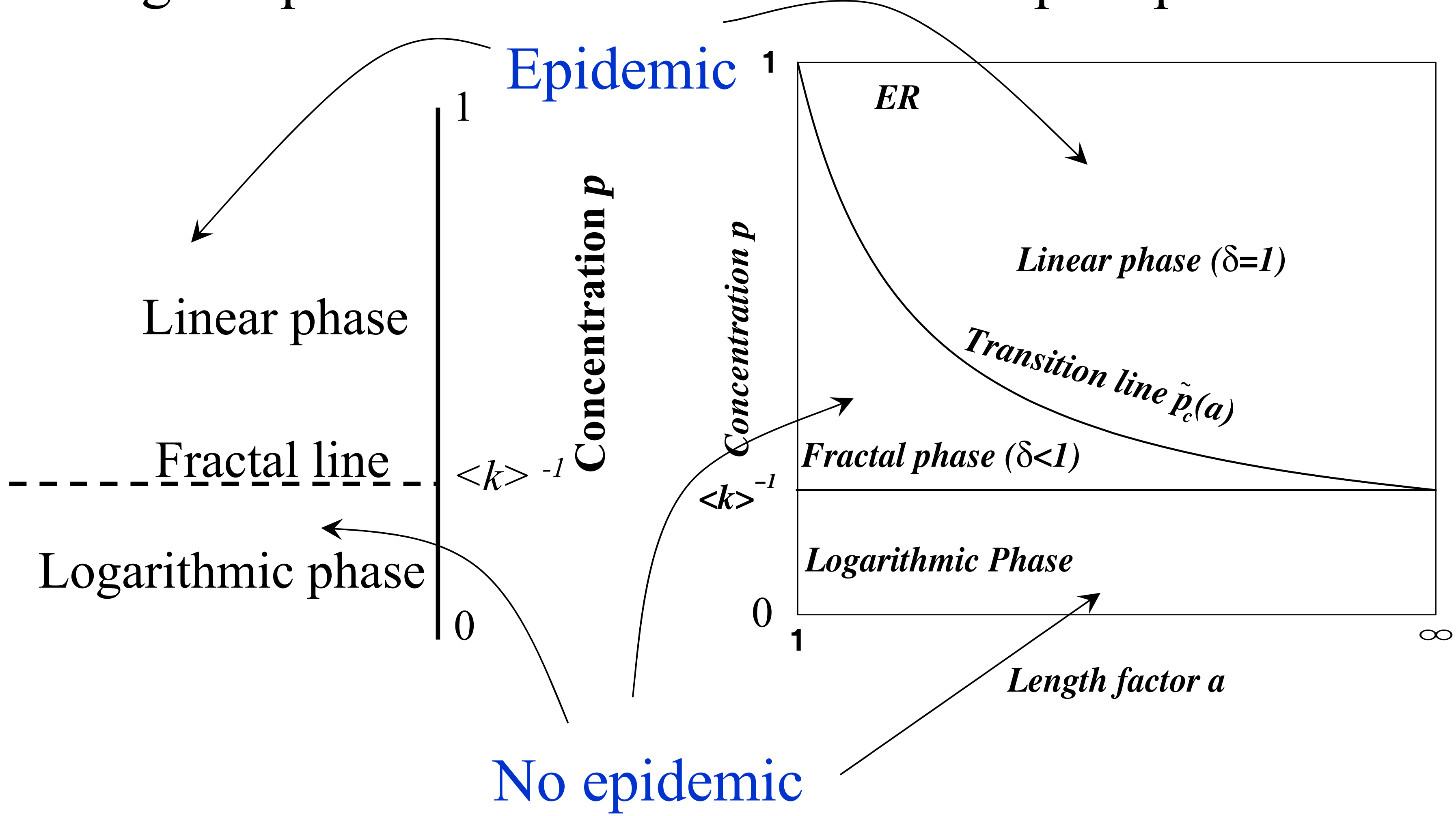




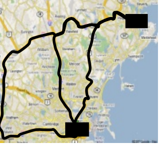
Comparison of phase diagram of regular &Limited Path Percolation (Erdős-Rényi)



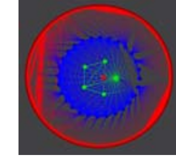
Regular percolation | Limited path percolation



Limited path percolation predicts a larger epidemic threshold.

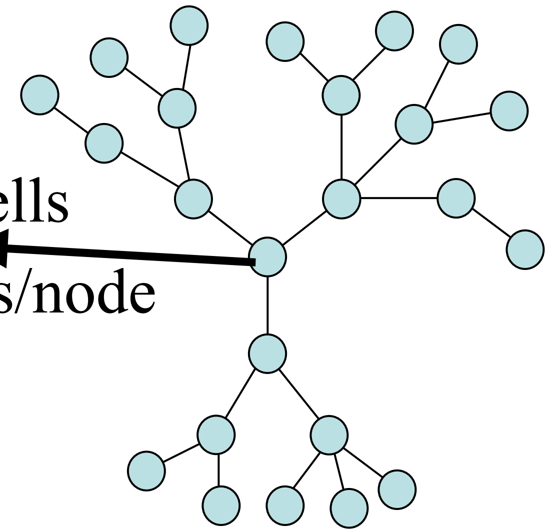


Theory: Erdős-Rényi network properties



- Nodes connected with branching factor κ_o:

  $1 + \kappa_o + \kappa_o(\kappa_o\text{-}1) + \kappa_o(\kappa_o\text{-}1)^2 \sim (\kappa_o\text{-}1)^l$



- Branching factor:

  $\kappa_o = \langle k^2 \rangle / \langle k \rangle = \langle k \rangle + 1$

- Typical distance l between nodes:

  $$l \sim \log N / \log(\kappa_o - 1) = \log N / \log\langle k \rangle$$

- Percolation threshold:

  $p_c = \langle k \rangle^{-1}$

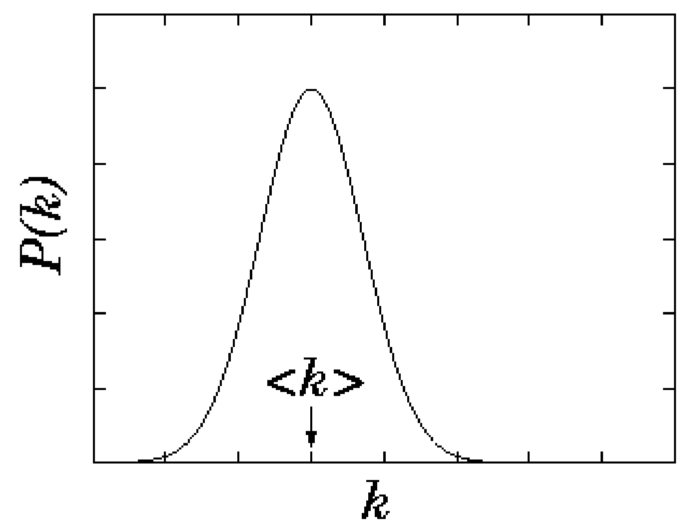
- Nodes connected at $p = p_c$:

  $S \sim N^{2/3}$

- Typical length at $p = p_c$:

  $l \sim N^{1/3}$

- Degree distribution:



New percolation model applied to complex networks

- Definition of connection: i and j are connected if $l'_{ij} \leq a l_{ij}$
- Notation:

$S_a(p)$: Largest cluster size at occupation p, length condition a

- Is there a critical occupation $p = \tilde{p}_c$ above which $S_a \sim N$?

Results: New limited path percolation transition

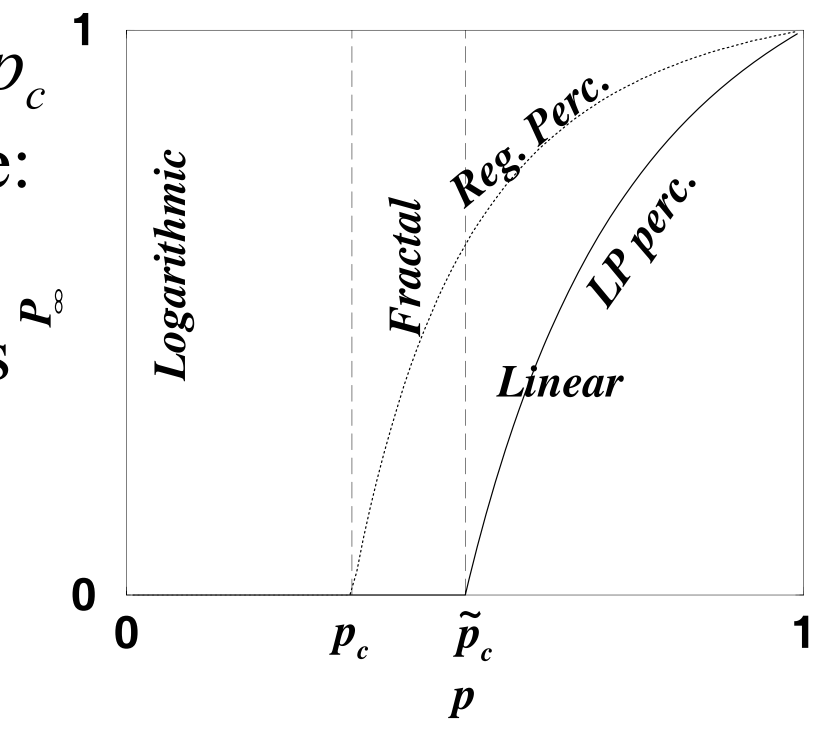
- Analytical scaling theory
- Find new critical occupation $\tilde{p}_c > p_c$
- Critical point is now a critical range:

$$S_a \sim N^{\delta}, \delta = \delta(a,p) \left(p_c < p < \tilde{p}_c\right)$$

- Below and above range, behavior is similar to regular percolation:

$$S_a \sim \log N \left(p < p_c\right)$$

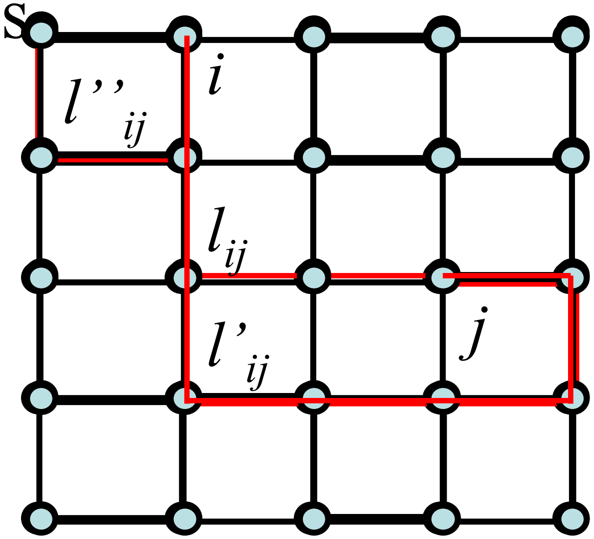
$$S_a \sim N \left(p > \tilde{p}_c\right)$$



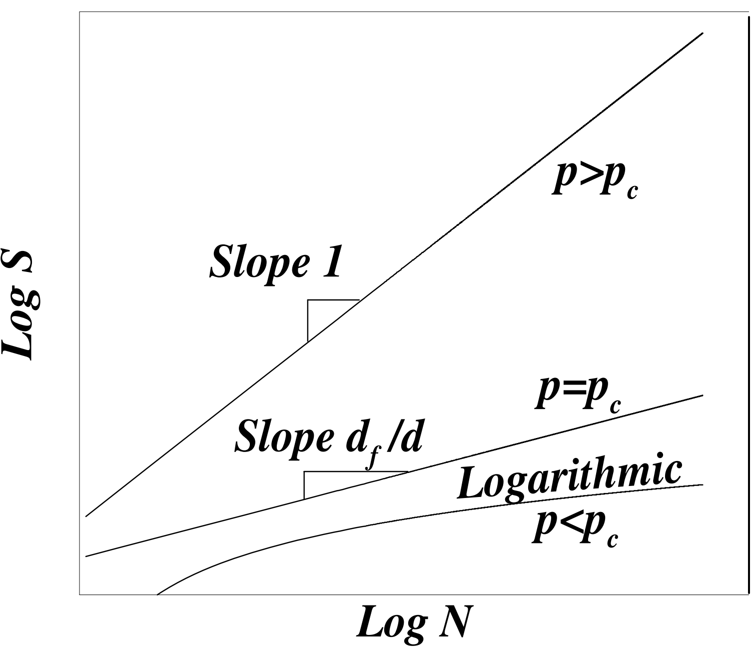
What is percolation theory?

Theory to determine connectivity in systems

p	i, j distance	$S(p)$: # connected nodes
$=1$	l_{ij}	N
<1	l'_{ij}	$P_\infty N$
	$l'_{ij} > l_{ij}$ due to removal	
$=p_c$	$l''_{ij} > l'_{ij}$	$N^{df/d}$
$<p_c$	most disconnec.	$\log N$

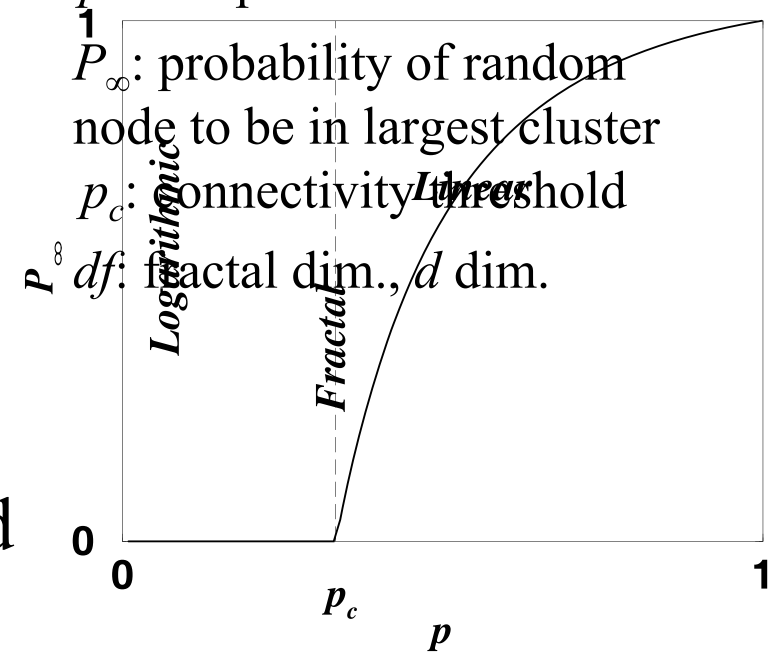


p: occupied fraction of links



Transition:

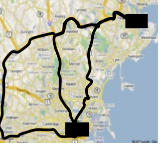
connected
↓
disconnected



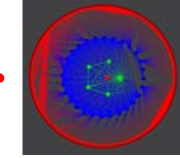
P_∞: probability of random node to be in largest cluster

p_c: connectivity threshold

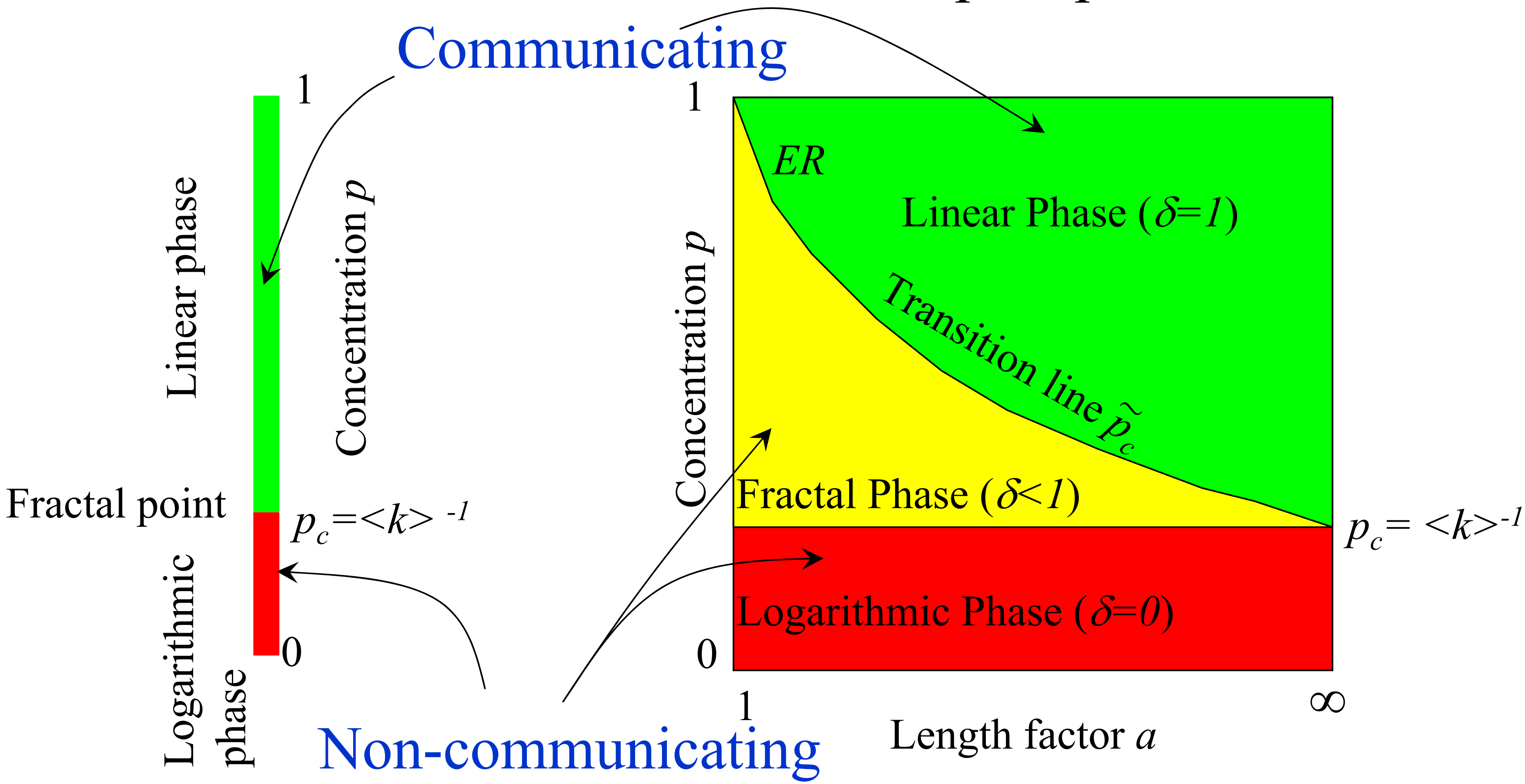
df: fractal dim., d dim.



Comparison of phase diagram of regular & Limited Path Percolation (Erdős-Rényi)



Regular percolation | Limited path percolation



Limited path percolation predicts a larger communication threshold.